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Mathematical Reviews

Vol. 8, No. 9 OCTOBER, 1947 Pages 497-556

HISTORY

Minoda, Takashi. On "Katuyô Sampô, book III" of Seki. III. Tôhoku Math. J. 49, 220–222 (1943). (Japanese)
For part II see the same J. 48, 167–173 (1941); these Rev. 7, 353.

Mikami, Yoshio. On Narikiyo Kuroda and surveying. Tôhoku Math. J. 49, 223–242 (1943). (Japanese)

***Gandz, Solomon. A few notes on Egyptian and Babylonian mathematics.** Studies and Essays in the History of Science and Learning Offered in Homage to George Sarton on the Occasion of his Sixtieth Birthday, 31 August 1944, pp. 449–462. Henry Schuman, New York, 1947.

***Cassina, Ugo. Sulla geometria egiziana.** Atti Secondo Congresso Un. Mat. Ital., Bologna, 1940, pp. 897–898. Edizioni Cremonense, Rome, 1942.
Cf. Period. Mat. (4) 22, 1–29 (1942); these Rev. 8, 1.

***Schrecker, Paul. On the infinite number of infinite orders. A chapter of the pre-history of transfinite numbers.** Studies and Essays in the History of Science and Learning Offered in Homage to George Sarton on the Occasion of his Sixtieth Birthday, 31 August 1944, pp. 359–373. Henry Schuman, New York, 1947.

***Rey, Abel. L'Apogée de la Science Technique Grecque. Les Sciences de la Nature et de l'Homme. Les Mathématiques d'Hippocrate à Platon.** Éditions Albin Michel, Paris, 1946. xviii+313 pp. 190 francs.
This is the first half of the fourth volume of the author's "La Science dans l'Antiquité." [For vol. 3 [1939] cf. these Rev. 1, 289.] The second part is to appear later. The whole manuscript is being published in the form left by its author at his death in 1939. The time limits indicated in the title are frequently transgressed, especially in the chapter on astronomy, which reaches from Plato to Hipparchus and beyond. The presentation of the interval is completely determined by the state of affairs where Tannery and Duhem left it. Many statements on details can be questioned.
O. Neugebauer (Providence, R. I.).

Marković, Željko. Sur la théorie de la mesure de Platon. Bull. Intern. Acad. Yougoslave. Cl. Sci. Math. Nat. 33, 1–25 (1940).
Discussion of Plato's number theory (the so-called "idea-numbers") and related topics in close connection with papers by O. Becker, Stenzel and Toeplitz, published in Quellen und Studien zur Geschichte der Mathematik.
O. Neugebauer (Providence, R. I.).

***Natucci, A. Rapida visione dei progressi della geometria elementare da Euclide ai nostri giorni.** Atti Secondo Congresso Un. Mat. Ital., Bologna, 1940, pp. 966–976. Edizioni Cremonense, Rome, 1942.

***Andrissi, G. L. Una nuova interpretazione del sistema di Filolao, più consona ai testi e che giustifica i dieci corpi mobili, e che mostra come in tale sistema la rotazione diurna degli astri dovesse considerarsi reale.** Atti Secondo Congresso Un. Mat. Ital., Bologna, 1940, pp. 903–907. Edizioni Cremonense, Rome, 1942.

***Andrissi, G. L. Gli antichi sistemi geocentrici e la confusione che generalmente regna attorno ad essi nelle storie delle scienze e della filosofia.** Atti Secondo Congresso Un. Mat. Ital., Bologna, 1940, pp. 908–911. Edizioni Cremonense, Rome, 1942.

***Andrissi, G. L. Una nuova interpretazione di alcuni brani di Platone che esclude in Platone ogni ipotesi sulla reale rotazione diurna della terra.** Atti Secondo Congresso Un. Mat. Ital., Bologna, 1940, pp. 912–920. Edizioni Cremonense, Rome, 1942.

***Andrissi, G. L. Valore e funzioni dell'astronomia medievale italiana nello sviluppo storico dell'astronomia e confutazione delle denigrazioni degli storici d'oltralpe.** Atti Secondo Congresso Un. Mat. Ital., Bologna, 1940, pp. 921–931. Edizioni Cremonense, Rome, 1942.

Procissi, A. Sui primi sistemi lineari, sulla "regula modi" di Cardano, e sul metodo di addizione di Buteone. Period. Mat. (4) 24, 141–151 (1946).

***Procissi, A. Di alcune lettere di Giovanni Ceva.** Atti Secondo Congresso Un. Mat. Ital., Bologna, 1940, pp. 895–896. Edizioni Cremonense, Rome, 1942.

***Varetti, Carlo Vittorio. Contributo alla storia dell'ottica nella prima metà del secolo XVII dal canocchiale di Galileo alle lenti del Torricelli.** Atti Secondo Congresso Un. Mat. Ital., Bologna, 1940, pp. 572–581. Edizioni Cremonense, Rome, 1942.

Tibiletti, Cesarina. Sul problema di Apollonio: la soluzione di Gaultier De Tours. Period. Mat. (4) 24, 152–161 (1946).

Karpinski, Louis Charles. Bibliographical check list of all works on trigonometry published up to 1700 A.D. Scripta Math. 12, 267–283 (1946).

***Agostini, A. La geometria degli infinitesimi di Gerolamo Saladini.** Atti Secondo Congresso Un. Mat. Ital., Bologna, 1940, pp. 886–894. Edizioni Cremonense, Rome, 1942.
Saladini lived 1735–1813.

***Dürr, Karl. Die Logistik Johann Heinrich Lamberts.** Festschrift zum 60. Geburtstag von Prof. Dr. Andreas Speiser, 47–65, Füssli, Zürich, 1945.

Dugas, René. Vicissitudes de la notion de force. Revue Sci. 84, 451–461 (1946).

Duarte, F. J. On the non-Euclidean geometries. Historical and bibliographical notes. Revista Acad. Colombiana Ci. Exact. Fís. Nat. 7, 63–81 (1946). (Spanish)

Reprinted from Estados Unidos de Venezuela. Bol. Acad. Ci. Fís. Mat. Nat. 9, 1–67 (1945); these Rev. 8, 2.

Bibliography of Professor C. Carathéodory. Bull. Soc. Math. Grèce 22, 198–207 (1946).

Castelnuovo, G. Commemorazione del socio Federigo Enriques. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 2, 3–21 (1947).

Obituary: Nikolaĭ Evgrafovič Kočin (1900–1945). Uspehi Matem. Nauk (N.S.) 1(11), no. 1, 27–29 (1946). (Russian)

Lyusternik, L. A. Obituary: Alekseĭ Nikolaevič Krylov (1863–1945). Uspehi Matem. Nauk (N.S.) 1(11), no. 1, 3–10 (1946). (Russian)

Amaldi, U. Commemorazione del socio Tullio Levi-Civita. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 1, 1130–1155 (1946).

Turnbull, H. W. Colin Maclaurin. Amer. Math. Monthly 54, 318–322 (1947).

***Turnbull, H. W. The Mathematical Discoveries of Newton.** Blackie & Son Limited, Glasgow, 1945. vii+68 pp. 5/–.

This is a handy summary of Newton's contributions to pure mathematics. Emphasis is placed on the development of his ideas. New information was obtained from unpublished material in the libraries of the universities of Edinburgh and of St. Andrews, an indication for the need for the publication of Newton's collected works in order to clarify the history of one of the most interesting periods in the development of mathematics. The author has wisely refrained from discussing the Leibniz-Newton controversy.

O. Neugebauer (Providence, R. I.).

Mangeron, Dumitru Ion. The scientific work of Gustav Magnus Mittag-Leffler. Revista Mat. Timişoara 26, 8 pp. (1946). (Romanian)

***Archibald, Raymond Clare. Material concerning James Joseph Sylvester.** Studies and Essays in the History of Science and Learning Offered in Homage to George Sarton on the Occasion of his Sixtieth Birthday, 31 August 1944, pp. 209–217 (4 plates). Henry Schuman, New York, 1947.

Hille, Einar. Jacob David Tamarkin—his life and work. Bull. Amer. Math. Soc. 53, 440–457 (1947).

ALGEBRA

on order

***Sierpiński, Wacław. Zasady Algebry Wyższej. [Principles of Higher Algebra].** With an appendix by Andrzej Mostowski: Outline of Galois Theory. Monografie Matematyczne, vol. 11. Warszawa-Wrocław, 1946. xii+437 pp. (Polish)

This textbook covers in a very thorough but elementary way the basic facts of classical algebra, leading up and into modern algebra. It stops short of ideal theory. The list of chapter headings with brief comments in parentheses should give an idea of the scope of the book. (I) Permutations, (II) Determinants, (III) Solution of linear equations, (IV) Linear transformations, (V) Matrices, (VI) Complex numbers, (VII) Proof of the fundamental theorem of algebra, (VIII) Polynomials (arithmetic of polynomials in the complex domain, interpolation formulae, decomposition of rational functions into simple fractions), (IX) Symmetric polynomials, (X) Equations of the 2d, 3d, and 4th degree, (XI) Equations of the division of the circle (roots of unity), (XII) Algebraic numbers (in the field of complex numbers), (XIII) Number fields (in the complex domain), (XIV) Impossibility proofs (trisection and similar topics), (XV) Systems of two algebraic equations, (XVI) Calculation of roots of algebraic equations (Sturm's and Newton's methods), (XVII) General theory of operations (abstract theory of binary operations), (XVIII) Substitutions, (XIX) Groups, (XX) Generalization of number fields (abstract fields).

The appendix by Mostowski gives a lucid and elementary account of Galois theory. The definition of the Galois group (separable case) is by means of substitutions on the roots of a polynomial. The fundamental theorem of Galois theory and its application to the solution of equations are treated.

S. Eilenberg (New York, N. Y.).

Joseph, A. W. A problem in derangements. J. Inst. Actuaries Students' Soc. 6, 14–22 (1946).

The problem solved is essentially the enumeration of permutations of n elements such that element i is in neither of positions i or $i+1$ ($i=1, \cdots, n-1$), element n not in position n. This is closely related to Lucas' problème des ménages: the number of ways of seating at a circular table n married couples, husbands and wives alternating, so that no wife is next to her own husband; indeed, apart from a factor $2n!$, it is precisely Lucas' problem with a straight table. The author's procedure is twofold: straightforward application of the method of inclusion and exclusion in terms of partial rencontres numbers $\Delta^p q!$ and derivation of the recurrence relation $u_n=(n-1)(u_{n-1}+u_{n-2})+u_{n-3}$. His first procedure leads incidentally to Schöbe's relation [Math. Z. 48, 781–784 (1943); these Rev. 5, 29] for allied numbers v_n such that $u_n=v_n+v_{n-1}$ in terms of the squares of rencontres numbers $\Delta^n 0!$. The recurrence relation is used to give a table of u_n, $n=1$ to 12, which also contains a comparison with the first 3 terms (all given) of the asymptotic expression

$$u_n \sim n!e^{-2}\left\{1-\sum_2^\infty(-)^i\frac{(i-1)(n-i)!}{i!n!}\right\}.$$

[Reviewer's note: the author seems to have missed the simplest expression for these numbers, namely

$$u_n=\sum(-)^i\binom{2n-i}{i}(n-i)!.$$

Also, it may be noted that $2nu_{n-1}=U_{n,1}$, $u_n-u_{n-1}=U_{n,0}$, where $U_{n,r}$ is a ménage number with r elements in forbidden positions; this is a simple consequence of relations among generating functions given by Kaplansky and Riordan, Scripta Math. 12, 113–124 (1946); these Rev. 8, 365.]

J. Riordan (New York, N. Y.).

Demelenne, J. Théorie des sommantes. Bull. Soc. Roy. Sci. Liége 15, 192–204 (1946).

This is a preliminary communication of a study of abstract elements suggested by the relation of cumulants and moments in statistical theory. Since the cumulants of the sum of independent variables are sums of cumulants of the variables, they suggest "suites sommantes," while the moments, whose generating functions are associated in powers, suggest "suites puissantes." To simplify the writing of the latter, a new notation is introduced according to which the binomial theorem is written

$$(x+y)_n = \frac{(x+y)^n}{n!} = \sum x_{n-k} y_k,$$

thus avoiding numerical coefficients here and in the multinomial theorem. [As noted by the author, this simplification has its cost in the complication of the ordinary formula for powers which in the new notation reads $x^p x^q = \binom{p+q}{p} x^{p+q}$.]

It is shown that there is a bi-unique relationship between the two kinds of sets of elements: if X_n is the nth element of a suite puissante, x_n of a suite sommante, then in the new notation

$$X_n = \sum x_1^{p_1} \cdots x_i^{p_i} \cdots x_n^{p_n}; \quad p_1 + 2p_2 + \cdots + np_n = n.$$

This is the formula for moments in terms of cumulants, shorn of coefficients by the new notation. The inverse expression is more complicated, and the author gives an algorithm which seems to the reviewer less simple than the symbolic relation $\kappa(\kappa+m)^i = m_{i+1}$ for cumulants (κ) and moments (m). Indeed the author seems to have missed the correspondent to the expression for cumulants in terms of moments. Finally applications of the theory to iterates, integrals and derivatives are shown; an application to the theory of random variables is promised. It is not clear to the reviewer what advantage the proposal has over the Blissard symbolic or umbral calculus, to which it bears a marked resemblance. *J. Riordan* (New York, N. Y.).

Rutherford, D. E. Some continuant determinants arising in physics and chemistry. Proc. Roy. Soc. Edinburgh. Sect. A. 62, 229–236 (1947).

The author evaluates a number of special continuants (determinants of special type) which occur in theoretical physics and chemistry. The evaluations are based on Wolstenholme's evaluation of the general continuant [see Muir, "The Theory of Determinants in the Historical Order of Development," vol. 4, Macmillan, London, 1923, p. 401] and the following theorem, which is proved. Set

$$L(M(x)) = L(0)\langle I\rangle + I\langle M(x)\rangle,$$

where $A\langle B\rangle$ is the direct product matrix usually denoted by $A \times \cdot B$. Theorem: if the matrix $L(x)$ is of the form $L(0)+xI$ and the matrix $M(x)$ is of the form $M(0)+xI$, then the latent roots of the matrix $L(M(0))$ are l_i+m_j; $i=1, \cdots, \lambda$; $j=1, \cdots, \mu$, where $l_1, \cdots, l_\lambda$ are the latent roots of $L(0)$ and $m_1, \cdots, m_\mu$ are the latent roots of $M(0)$.

G. B. Price (Lawrence, Kan.).

Massonnet, Charles. Sur une condition suffisante pour qu'un déterminant soit positif. Bull. Soc. Roy. Sci. Liége 14, 313–317 (1945).

The author proves the following theorem. A determinant $|a_{ij}|$ is nonnegative if each of its diagonal elements a_{ii} is positive and equal to or greater than the sum of the absolute values of the other elements in the same row. Corollary: a determinant $|a_{ij}|$ of order n is nonnegative if the product of its diagonal elements is positive and $\sum_{j\neq i}|a_{ij}|/[|a_{ii}|\,|a_{jj}|] \leqq 1$ for $i=1, 2, \cdots, n$. [The theorem, in various forms and generalizations, has been treated many times before; the method of proof is not new. References: Lévy, C. R. Acad. Sci. Paris 93, 706–708 (1881); Desplanques, J. Math. Spéc. (3) 1, 12–13 (1887); Minkowski, Nachr. Ges. Wiss. Göttingen. Math. Phys. Kl. **1900**, 90–93; Hadamard, Leçons sur la Propagation des Ondes et les Équations de l'Hydrodynamique, Paris, 1903, pp. 13–14; Rohrbach, Jber. Deutsch. Math. Verein. 40, 49–53 (1931); Furtwängler, Akad. Wiss. Wien, S.-B. IIa. **1936**, 527–528; Ostrowski, Bull. Sci. Math. (2) 61, 19–32 (1937); Comment. Math. Helv. 10, 69–96 (1937); Parodi, C. R. Acad. Sci. Paris 223, 23–25 (1946); these Rev. 8, 128.] *G. B. Price* (Lawrence, Kan.).

Lepage, Th.-H. Sur certaines congruences de formes alternées. Bull. Soc. Roy. Sci. Liége 15, 21–31 (1946).

Let $P=(p_{ij})$ be a symmetric matrix $(p_{ij}=p_{ji};\ i, j=1, \cdots, n)$, E the $n\times n$ unit matrix, and let $F(p_{ij})$ be a function linear in the determinants of order n of the rectangular matrix (E, P). The main result of the paper is the following theorem on symbolic (alternées) forms in the $2n$ indeterminates $x_1, \cdots, x_n;\ y_1, \cdots, y_n$. Let $\Gamma = x_1y_1 + \cdots + x_ny_n$, $w_i = y_i - p_{ij}x_j$; then in the family of symbolic forms (Ω) of degree n satisfying the equation $\Omega w_1 \cdots w_n = F(p_{ij})\Gamma^n/n!$, there exists a unique form Ω independent of the p_{ij} and such that $\Omega\Gamma = 0$. *F. G. Dressel* (Durham, N. C.).

Richardson, A. R. The composition of cubic forms. Duke Math. J. 14, 27–30 (1947).

The author indicates a method for studying the composition of n-ary cubic forms which is related to his study of the class-rings in multiplicative systems [Ann. of Math. (2) 44, 21–39 (1943); these Rev. 4, 185]. Applying the method in the binary case, he proves that every binary cubic form, over a commutative and associative ring R having a modulus 1, which represents 1 over R, arises by triplication.

R. Hull (Seattle, Wash.).

Abstract Algebra

Taussky, Olga, and Todd, John. Some aspects of modern algebra. Science Progress 138, 253–268 (1947).

Zariski, Oscar. A new proof of Hilbert's Nullstellensatz. Bull. Amer. Math. Soc. 53, 362–368 (1947).

Der Verfasser analysiert die Beweise für den Hilbertschen Nullstellensatz. Dieser lässt sich bekanntlich darauf zurückführen, dass ein Integritätsbereich, der endlich über einem Körper ist, genau dann ein Körper ist, wenn er algebraisch über einem Körper ist. Diese Tatsache wird neu bewiesen, unabhängig vom Begriff der Ganz-Abhängigkeit. Ebenso wird darüber hinaus bewiesen, dass ein Integritätsbereich, der endlich über zwei Körpern K_1 und K_2 ist, denselben Transzendenzgrad über K_1 und K_2 hat. *P. Lorenzen.*

Ballieu, Robert. Anneaux finis; systèmes hypercomplexes de rang deux sur un corps. Ann. Soc. Sci. Bruxelles. Sér. I. 61, 117–126 (1947).

The author shows that any finite ring is a direct sum of rings of prime power order and determines the latter when

the additive group is cyclic. [These results were given by Vandiver, Trans. Amer. Math. Soc. **13**, 293–304 (1912).] He also finds all rings whose additive group is of type (p, p). This is the same as finding all algebras of order two over $GF(p)$. The known result, which goes back to Cayley [Proc. London Math. Soc. (1) **15**, 185–197 (1883)], is recapitulated without references. *I. Kaplansky* (Chicago, Ill.).

Schwarz, Štefan. A hypercomplex proof of the Jordan-Kronecker's "principle of reduction." Časopis Pěst. Mat. Fys. **71**, 17–20 (1946). (English. Czech summary)

Let $f(x)$ and $g(x)$ be two irreducible separable polynomials over a field P and let each be factored in the field obtained by adjoining to P a zero of the other: $f(x)=f_1(x,\beta)\cdots f_r(x,\beta)$, $g(x)=g_1(x,\alpha)\cdots g_s(x,\alpha)$, where α is a root of $f(x)=0$ and β a root of $g(x)=0$. By expressing the ring $\mathrm{P}(\alpha)[x]/(g(x))$ as a direct sum of fields, the author gives a short proof of Kronecker's theorem that $r=s$ and, under proper numbering,

$$(\text{degree } f_i)/(\text{degree } g_i)=(\text{degree } f)/(\text{degree } g)$$

for each i. He also proves a theorem of Loewy [Math. Z. **15**, 261–273 (1922)] which gives further relations between these factors. *G. Whaples* (Bloomington, Ind.).

Whaples, George. On a conjecture about infinite class fields. Bull. Amer. Math. Soc. **53**, 377–380 (1947).

On sait que deux extensions galoisiennes de degré fini d'un corps de nombres algébriques de degré fini sont isomorphes quand toutes leurs extensions locales le sont. L'auteur démontre que le résultat analogue est faux pour les extensions algébriques galoisiennes de degré infini de tels corps, même quand ces extensions sont abéliennes. L'auteur définit l'extension locale d'une extension algébrique galoisienne de degré infini K/k pour une valuation donnée du corps de base comme l'extension du complété $\bar{k}$ de k par rapport à cette valuation qui est le corps composé $\bar{K}$ de K et de $\bar{k}$ (ce corps composé est unique à l'isomorphie par rapport à $\bar{k}$ près). *M. Krasner* (Paris).

Crosby, W. J. R. Generic algebras. Amer. J. Math. **69**, 333–347 (1947).

Let S be a central simple algebra over a field P and suppose the elements of a factor set for S lie in P. The author constructs a related algebra Σ over a field Π which is an algebraic function field over P; Σ is called a generic algebra. It is shown that certain problems in the theory of algebras can be reduced to the case of generic algebras. A notable example is the question as to whether there exist central division algebras which are not crossed products. *I. Kaplansky* (Chicago, Ill.).

THEORY OF GROUPS

Foulkes, H. O. Irreducible matrix representations of certain finite groups. J. London Math. Soc. 21, 226–233 (1946).

It is proved that the groups of order p^2q have all their irreducible representations transformable to monomial form. The same was already known for the groups of orders pq, pqr and p^3. Explicit forms are given for the irreducible monomial representations of the groups of all these orders directly in terms of their generating relations and it is shown how in certain cases these representations can be transformed into the field of the characters. [Line 2, p. 232, should read ". . . order pq^2 . . ." instead of ". . . order p^2q."] *R. M. Thrall* (Ann Arbor, Mich.).

Levi, F. W. The ring of endomorphisms for which every subgroup of an Abelian group is invariant. J. Indian Math. Soc. (N.S.) **10**, 29–31 (1946).

If A is an additive Abelian group, then $R(A)$ is the ring of all the endomorphisms f of A which satisfy $Sf\leqq S$ for every subgroup S of A. The author proves the following results. (a) If A contains elements of order 0, then $R(A)$ is the ring of integers. (b) If A does not contain elements of order 0, then A is the direct sum of its primary components $A(p)$ and the ring $R(A)$ is the direct sum of the rings $R(A(p))$. (c) If A is an Abelian p-group, then either there exists a least upper bound p^m of the orders of the elements in A and $R(A)$ is the ring of integers modulo p^m, or else $R(A)$ is the ring of p-adic integers. *R. Baer.*

Grayev, M. Structural isomorphisms of topological Abelian groups. Rec. Math. [Mat. Sbornik] N.S. **20(62)**, 125–144 (1947). (Russian. English summary)

For $i=1, 2$ let G_i denote a separable locally compact Abelian group and let L_i denote the lattice of closed subgroups of G_i. The author studies conditions on G_1 under which every lattice isomorphism between L_1 and L_2 is induced by some topological group isomorphism between G_1 and G_2. In addition he studies the somewhat weaker conditions which suffice when only those lattice isomorphisms preserving a certain topology are considered. One further restricted class of lattice isomorphisms is also briefly treated. The principal results are as follows. Let d denote the dimension of G_1, let r denote its rank (rank is defined here in such a way that it turns out to be the dimension of the dual group) and let v be the dimension of its vector space component. Then if any one of the following conditions is fulfilled every lattice isomorphism between L_1 and L_2 is induced by exactly two topological group isomorphisms between G_1 and G_2: (a) $d=0$ and $r\geqq 2$, (b) $r=0$ and $d\geqq 2$, (c) $v\geqq 2$. Moreover, if either (a) $v\geqq 1$ or (b) $d+r\geqq 2$ then every lattice isomorphism between L_1 and L_2 which preserves the topology alluded to above is induced by exactly two topological group isomorphisms between G_1 and G_2. Examples are given showing that these results are the best possible of their kind. The proofs depend upon the Pontrjagin-van Kampen structure and duality theorems for locally compact Abelian groups and on the results of a corresponding study of discrete groups made by Baer [Amer. J. Math. **61**, 1–44 (1939)]. *G. W. Mackey.*

Raikov, D. On the completion of topological groups. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **10**, 513–528 (1946). (Russian. English summary)

The author gives detailed results about the completion of an arbitrary topological group. The basic concept is that of a "funnel," defined to be a family of pairwise overlapping sets which includes sets arbitrarily small on the right (that is, sets M such that MM^{-1} is part of an arbitrarily prescribed neighborhood U of the group-identity) and sets arbitrarily small on the left (that is, such that $M^{-1}M$ is part of U). To make the system of all funnels into a group it suffices to define the product of two funnels as the family of all sets MN, where M is in the first and N in the second funnel, and to introduce at the same time an

equivalence relation holding between two funnels if and only if the group-identity belongs to the closure of every set MN^{-1}, where M is in the first and N in the second funnel. A funnel is said to converge to an element g of the group if g is in the closure of every set belonging to the funnel. The group of all funnels contains the system of all convergent funnels as a subgroup isomorphic (in the obvious way) to the given group. A group is said to be complete if all its funnels are convergent. In a certain natural topology the group of all funnels is complete. An open basis for this topology is specified to consist of the funnel-systems O^* each of which is obtained in terms of some open subset O of the group as the system of all funnels having members strongly contained in O, a set A being strongly contained in a set B when for some neighborhood U of the group-identity $UAU \subset B$. The complete topological groups are identified with the absolutely closed topological groups, defined by A. D. Alexandroff [C. R. (Doklady) Acad. Sci. URSS **37**, 118–121 (1942); these Rev. **5**, 45] as those which are contained as everywhere dense subgroups in no proper extension. It is also shown that if an arbitrary topological group G is contained as an everywhere dense subgroup in an absolutely closed topological group then the latter is algebraically and topologically isomorphic to the funnel-group of G by a correspondence pairing the elements of G with the convergent funnels in the natural manner.

M. H. Stone (Chicago, Ill.).

Riss, Jean. Représentations continues des groupes topologiques abéliens dans le groupe additif des nombres réels. C. R. Acad. Sci. Paris **224**, 987–988 (1947).

The author studies the connection between the representations of a given topological Abelian group G, without elements of finite order, into the group R of the real numbers under addition, on the one hand, and the "convex" and related subsets of G, on the other hand. For the case when G is the additive group of a topological linear space a similar connection has been used by G. W. Mackey [Proc. Nat. Acad. Sci. U. S. A. **29**, 315–319 (1943); **30**, 24 (1944); Trans. Amer. Math. Soc. **60**, 519–537 (1946); these Rev. **5**, 99; **8**, 519]. A "convex" subset A is one such that $x^n \epsilon A^n$ implies $x \epsilon A$, and any subset S is contained in a smallest convex set $k(S)$. A "semi-group" is a subset T of $G-\{e\}$ (e being the group identity) which is maximal relative to the property $T^2 \subset T$. Various results are stated, notably: (1) a semi-group T of G induces, in a well-defined fashion, a representation of G into R, which is continuous if T contains an interior point, and every representation of G into R can be obtained in this way; (2) G can be mapped in continuous and one-to-one fashion into a subgroup of a topological product of real number groups if and only if $\cap_V k(V) = \{e\}$, V being a general neighborhood of e.

I. E. Segal (Chicago, Ill.).

Riss, Jean. Sur les représentations réelles des groupes topologiques abéliens. C. R. Acad. Sci. Paris **224**, 1095–1097 (1947).

This is a continuation of the paper reviewed above and uses concepts introduced there. In terms of these concepts the author studies the problem of extending a representation defined on a subgroup and states a condition for a representation to be a homomorphism (the latter term is not defined).

I. E. Segal (Chicago, Ill.).

Kawada, Yukiyosi. On the probability distribution on a compact group. II. Proc. Phys.-Math. Soc. Japan (3) **23**, 669–686 (1941).

[For part I, by Kawada and Itô, see the same Proc. (3) **22**, 977–998 (1940); these Rev. **2**, 223.] On a compact separable connected group G, let $m_G(A)$ be the Haar measure with $m_G(G)=1$ and let $p(A)$ be any other σ-additive positive set function with $p(G)=1$. The convolution $p*q$ is defined as usual. If a group-invariant distance $\rho(a, b)$ is defined on G, then for each $l>0$ the author introduces $Q(p, l) = \sup p(sV_l)$ for $s \epsilon G$, where V_l is the sphere of radius l and center at unity on G. For the one-dimensional torus, $Q(p, l)$ was produced by P. Lévy, who proved that $Q(m_G, l) \leqq Q(p, l)$ and, if for all l, except countably many, equality holds, then $p = m_G$. The author proves this for his general case. For Abelian G the author furthermore proves among others the following result. Corresponding to any sequence $p_1, p_2, p_3, \cdots$ there exists a closed subgroup H of G such that $\lim_{m\to\infty} \lim_{n\to\infty} Q(P_m \times P_{m+1} \times \cdots \times P_n, l) = Q(m_H, l)$, where $m_H(A)$ is the set function of G which on subsets of H is the Haar measure of H.

S. Bochner.

Liapin, E. Free systems with an infinite univalent operation. C. R. (Doklady) Acad. Sci. URSS (N.S.) **51**, 493–496 (1946).

Multiplicative systems, in which words of transfinite length may occur, are considered. If Ξ is a class of such systems, $S \epsilon \Xi$ and $A \subset S$, then A is said to be a free complex with respect to the class Ξ if, for every $R \epsilon \Xi$, every map $A \to R$ can be extended to a homomorphism $S \to R$. If the free complex A generates S then S is said to be free with respect to Ξ. Several propositions concerning these concepts are stated without proof.

S. Eilenberg.

NUMBER THEORY

Pipping, Nils. Tafel der Diagonalkettenbrüche für die Quadratwurzeln aus den natürlichen Zahlen von 1–500. Acta Acad. Aboensis **15**, no. 10, 11 pp. (1947).

By a diagonal (Minkowski) continued fraction expansion of a number ξ is meant that semiregular continued fraction

$$\xi = b_0 + \frac{\epsilon_1}{b_1} + \frac{\epsilon_2}{b_2} + \cdots, \qquad \epsilon_i^2 = 1,$$

for which $|\xi - A_k/B_k| < \frac{1}{2} B_k^{-2}$, where A_k/B_k is the kth convergent. In case $\xi = D^{\frac{1}{2}}$, the square root of a positive integer, then the expansion is periodic, the partial quotients b_i being palindromic, as in the case of the regular continued fraction. The table gives the b's and ϵ's in the periods for every $D \leqq 500$. Some properties of these elements are observed from this tabular evidence, such as the palindromic character of the ϵ's which the author promises to prove in a later paper. Of the 478 cases tabulated only **144** are regular.

D. H. Lehmer (Berkeley, Calif.).

Sarantopoulos, Spyridion. Quelques théorèmes sur les nombres entiers. Bull. Soc. Math. Grèce **21**, 1–33 (1941).

This is a continuation of a paper begun in the same Bull. **20**, 85–100 (1940); these Rev. **2**, 34. The first part consists of many rather complicated theorems on divisors of $x^u \pm y^\lambda$ and similar polynomials, of which the following is typical. Let λ be an odd prime and assume that x and y are relatively prime; if $x^u \pm y^\lambda$ divides $xs^\lambda \pm \epsilon$ but not $s^{-\theta}[xs^\lambda \pm (\mp\epsilon)^{u-1} ys^u]$ (where θ is the smaller of the numbers u, λ), then either λ

divides $x^u \pm y^\lambda$, or $x^u \pm y^\lambda$ contains a prime factor of the form $p=2\lambda r+1$, or both of these things happen. Various special cases are considered, some of these being well-known theorems. In the second part of the paper the following theorem is proved. If A and b are two integers prime to M and u is prime to $\varphi(M)$, then A can be represented uniquely in the form $A=b\epsilon^u+M\beta$, where $0<\epsilon<M$. Several applications on the representation of numbers in various forms are given. *H. W. Brinkmann* (Swarthmore, Pa.).

van der Blij, F. On the theory of simultaneous linear and quadratic representation. I, II, III. Nederl. Akad. Wetensch., Proc. **50**, 31–40, 41–48, 166–172 = Indagationes Math. **9**, 16–25, 26–33, 129–135 (1947).

The author determines the numbers of solutions of the following pairs of simultaneous Diophantine equations: (1) $x_1^2+x_2^2+\cdots+x_r^2=n$, $x_1+x_2+\cdots+x_r=m$ in the cases $r=3, 4, 5, 8$; (2) $s_1x_1^2+s_2x_2^2+s_3x_3^2=n$, $s_1x_1+s_2x_2+s_3x_3=m$ in the cases (a) $s_1=s_2=1$, $s_3=2$, (b) $s_1=s_2=1$, $s_3=3$, (c) $s_1=1$, $s_2=s_3=2$, (d) $s_1=1$, $s_2=s_3=3$, (e) $s_1=3$, $s_2=2$, $s_3=1$; (3) $9x_1^2+x_2^2+x_3^2+x_4^2=n$, $9x_1+x_2+x_3+x_4=m$. *T. Estermann* (London).

van der Blij, F. On the theory of simultaneous linear and quadratic representation. IV, V. Nederl. Akad. Wetensch., Proc. **50**, 298–306, 390–396 = Indagationes Math. **9**, 188–196, 248–254 (1947).

[Cf. the preceding review.] The author determines the number of solutions of the simultaneous Diophantine equations $x_1^2+x_2^2+\cdots+x_r^2=n$, $x_1+x_2+\cdots+x_r=m$, for $r=5, 6$ and 7, and shows that the analogous result for $r=9$ is not true. He also obtains formulae for the number of representations of a number as the sum of (i) three octagonal numbers, (ii) three pentagonal numbers, (iii) four triangular numbers. He determines the numbers of solutions of certain cubic Diophantine equations, e.g., $x^3+y^3+z^3-3xyz=N$. *T. Estermann* (London).

Schmid, F. Über die Gleichung $x^3+y^3+z^3=0$. Akad. Wiss. Wien, S.-B. IIa. **152**, 7–14 (1944).

The author gives a proof of the impossibility in rational integers of the equation of the title, based on the theorem that the discriminant of a cubic algebraic number field is divisible by at least one rational prime. Since Kronecker proved that the impossibility of the cubic Fermat equation implies the cubic case of the Minkowski discriminant theorem, the essential equivalence of the former with the latter follows. *R. Hull* (Seattle, Wash.).

***Mordell, L. J. A Chapter in the Theory of Numbers.** Cambridge, at the University Press; New York, The Macmillan Company, 1947. 31 pp. $.40.

This inaugural lecture is an interesting historical account of the Diophantine equation $y^2=x^3+k$. *H. W. Brinkmann* (Swarthmore, Pa.).

***Mordell, L. J. Geometry of numbers.** Proc. First Canadian Math. Congress, Montreal, 1945, pp. 265–284. University of Toronto Press, Toronto, 1946. $3.25.

Report on recent work in the geometry of numbers, especially on various results obtained by Davenport, Mahler, Mordell and Ollerenshaw. Most of the results and methods reported concern domains which are not convex. *V. Jarník* (Prague).

Oppenheim, A. Two lattice-point problems. Quart. J. Math., Oxford Ser. **18**, 17–24 (1947).

The author considers the class $E(f)$ of ellipses which have center O and, as a pair of conjugate diameters, the given pair of straight lines $f(x, y)\equiv ax^2+2hx+by^2=0$, where $h^2>ab$, $D=2(h^2-ab)^{\frac{1}{2}}$ and f has no rational factors. Let Δ_0 be the lower bound of the area of those ellipses of $E(f)$ which have a lattice point on the boundary and no lattice points inside (except O); Δ_1 is defined analogously, but now the ellipses must have at least two lattice points P, Q (not collinear with O) on the boundary and none inside. It is shown that Δ_0 equals the lower bound of $2\pi|f(x, y)|D^{-1}$ for integral x, y (not both zero), whence $\Delta_0\leqq 2\pi/5^{\frac{1}{2}}$. The problem of finding Δ_1 turns out to be equivalent to that of the lower bound $\mathfrak{M}(C_f)$ of $|(A+B)/2H|$, as $Ax^2+2Hxy+By^2$ runs through all forms unimodularly equivalent to f. Using continued fractions, the author shows that $\mathfrak{M}\leqq\frac{1}{2}$; $\mathfrak{M}=\frac{1}{2}$ only if either $f\sim c(x^2+2xy-2y^2)$ or $f\sim c(x^2+4xy-3y^2)$. Now it is readily proved that $\Delta_1=\pi/(1-\mathfrak{M}^2)^{\frac{1}{2}}$ and hence $\pi\leqq\Delta_1\leqq 2\pi/3^{\frac{1}{2}}$. *N. G. de Bruijn* (Delft).

Davenport, H. Sur une extension d'un théorème de Minkowsky. C. R. Acad. Sci. Paris **224**, 990–991 (1947).

(I) Let $Q(x, y, z)$ be an indefinite ternary quadratic form with the discriminant D. Let x_0, y_0, z_0 be arbitrary real numbers. Then there are three integers x, y, z such that

$$(1)\qquad |Q(x+x_0, y+y_0, z+z_0)|\leqq k|D|^{\frac{1}{3}},$$

where $k^3=.27$. (II) If (2) $Q=x^2+5y^2-z^2+5yz+zx$, $x_0=y_0=\frac{1}{2}$, $z_0=0$, it is impossible to satisfy the inequality (1) in the strict sense (i.e., with $<$ instead of $\leqq$). (III) There exists an absolute constant $k'<k$ so that k in (1) may be replaced by k' if we exclude forms $\alpha Q_0(x, y, z)$, where α is a real number and Q_0 is equivalent to (2). The proofs will appear elsewhere. The following lemma is emphasized by the author. Let α, β, γ, δ, c_1, c_2 be real numbers, $\alpha\delta-\beta\gamma=\Delta\neq 0$, $\mu>0$, $\nu>0$, $\mu\nu\geqq\frac{1}{16}$. Then there exist integers x, y such that $-\nu|\Delta|\leqq(\alpha x+\beta y+c_1)(\gamma x+\delta y+c_2)\leqq\mu|\Delta|$. [If $\mu=\nu$, this is a well-known analogue of (1) for binary forms; see, e.g., Koksma, Diophantische Approximationen, Ergebnisse der Math., v. 4, no. 4, Springer, Berlin, 1936, pp. 18–20.] *V. Jarník* (Prague).

Kotzig, Anton. Sur les "translations k." Časopis Pěst. Mat. Fys. **71**, 55–66 (1946). (Czech. French summary)

Let n, k be given integers with $1\leqq k\leqq n$. A "translation k" of a point $(x_1, \cdots, x_n)$ in n-dimensional space is one which replaces it by $(x_1-a_1, \cdots, x_n-a_n)$, where $a_1, \cdots, a_n$ are nonnegative integers and the number l of a's which are different from zero satisfies $1\leqq l\leqq k$. Let M denote the set of all points $(x_1, \cdots, x_n)$ for which $x_1, \cdots, x_n$ are nonnegative integers. The author proves that there is a unique subset A of M such that (1) any point of M can be changed into a point of A by a suitable translation k; (2) no point of A can be changed into another point of A by a translation k. The explicit construction of A is that it consists of all points $(x_1, \cdots, x_n)$ for which $b_m(x_1)+\cdots+b_m(x_n)\equiv 0 \bmod (k+1)$, where $b_m(x)=0$ or 1 denotes the coefficient of 2^m in the dyadic expansion of a nonnegative integer x. *H. Davenport* (Stanford University, Calif.).

Selberg, Sigmund. On the distribution of the positive integers of the form $pp_1^{\alpha_1}p_2^{\alpha_2}\cdots p_n^{\alpha_n}$. Norske Vid. Selsk. Forh., Trondhjem **16**, no. 24, 87–90 (1943).

This paper contains a proof of a formula for the number of integers $m\leqq x$ for which $m=pp_1^{\alpha_1}p_2^{\alpha_2}\cdots p_n^{\alpha_n}$, where

$p, p_1, p_2, \cdots, p_n$ are distinct primes. The proof is made by induction on n from the case $n=0$ (the prime number theorem). Applications are made to the number of integers represented as a product of n primes, not necessarily distinct, and to the number of integers which have exactly n divisors. *R. D. James* (Vancouver, B. C.).

Selberg, Sigmund. On the sum $\sum_{m\leq x} \frac{1}{m}$, where m is of a given standard form. Norske Vid. Selsk. Forh., Trondhjem 16 (1943), no. 27, 99–102 (1944).

Let n be a positive integer and 2^ν the highest power of 2 dividing n. Let $S_n(x)=\sum 1/m$, summed over $1\leqq m\leqq x$ and $d(m)=n$, and $w_n(x)=\sum 1$ over the same range. The author proves, for any $q>0$, $S_n(x)=x^{-1}w_{2n}(x)\log x+O(\log\log^{-q}x)$ and

$$S_n(x)=\sum_{k=1}^{\nu+1}\frac{c_k}{(\nu-k+1)!}(\log\log x)^{\nu-k+1}+O(\log\log^{-q}x),$$

where $c_1, \cdots, c_{\nu+1}$ depend on $2^{-\nu}n$ only. These results are deduced from two previous papers by the author [Skr. Norske Vid. Akad. Oslo. I. **1942**, no. 5; these Rev. **6**, 57, and the paper reviewed above]. *H. Heilbronn* (Bristol).

Selmer, Ernst S. Über Primzahlen von der Form x^2+1. Norske Vid. Selsk. Forh., Trondhjem 15 (1942), no. 39, 149–152 (1943).

Hardy and Littlewood [Acta Math. **44**, 1–70 (1922)] conjectured an asymptotic formula for $P(n)$, the number of primes of the form x^2+1 less than n. In this paper a similar formula for primes of the form x^2+k is stated and verified for $n=2981^2$ and $k=1, 2, \cdots, 10$. With one or two exceptions the agreement between the calculated value and the actual count is surprisingly good considering the many number-theoretic tables involved in the computation.
R. D. James (Vancouver, B. C.).

Kuhn, Pavel. Zur Viggo Brun'schen Siebmethode. I. Norske Vid. Selsk. Forh., Trondhjem 14 (1941), no. 39, 145–148 (1942).

Using a result of Tartakowski [C. R. (Doklady) Acad. Sci. URSS (N.S.) **23**, 121–125, 126–129 (1939); the author's reference is incorrect] based on the Brun sieve method, the author proves the following theorem. For sufficiently large x there are $2.3\ e^{-C}x^{\frac{1}{2}}/\log x$ numbers between x and $x+x^{\frac{1}{2}}$ which are divisible by at most four primes. Here C is the Euler constant. *R. D. James* (Vancouver, B. C.).

Kuhn, Pavel. Zu den Mittelwerten zahlentheoretischer Funktionen. Norske Vid. Selsk. Forh., Trondhjem 14 (1941), no. 42, 157–160 (1942).

In this paper the integral

$$\int_0^x dx_1\int_0^{x_1}dx_2\cdots\int_0^{x_{m-1}}[x_m/n]dx_m$$

is expressed in terms of Bernoulli polynomials of order $m+1$. The proof is made by induction on m and the result is applied to the function $A(x, u)=[x/1]+[x/2]+\cdots+[x/u]$, where $u=x^{\frac{1}{2}}$. *R. D. James* (Vancouver, B. C.).

Kuhn, Pavel. Zur elementaren Abschätzung des Mittelwertes der Dirichletschen Teilerfunktion. Norske Vid. Selsk. Forh., Trondhjem 15, no. 8, 29–32 (1942).

Let $D(x)$ denote Dirichlet's divisor function and write $D(x)=x\log x+(2C-1)+E(x)$, where C is the Euler constant. Brun [Skr. Norske Vid. Akad. Oslo. I. **1941**, no. 12 (1942); these Rev. **7**, 275] found an explicit upper bound for $|x^{-1}\int_0^x E(t)dt|$. In this note a refinement of Brun's method leads to a smaller bound. *R. D. James.*

Carlitz, L., and Cohen, Eckford. Divisor functions of polynomials in a Galois field. Duke Math. J. 14, 13–20 (1947).

Let $M=M(x)$, Z, A, B be polynomials with coefficients in $GF(p^n)$, leading coefficients 1. For $\deg M=2ek$, $k\geqq 0$, $e>0$, the authors define

$$\delta_s^e(M)=\begin{cases}\displaystyle\sum_{Z^e|M}^{\deg Z=s} 1, & s\geqq 0,\\ 0, & s<0,\end{cases}$$

$$\gamma_s^e(M)=\delta_s^e(M)-p^{n(1-e)}\delta_{s-1}^e(M),$$

$$\rho_s^e(M, \mu)=p^{kn[s(e-1)+e+1]}\sum_{z=0}^{k}\mu^z p^{nz(e-s-1)}\gamma_z^e(M),$$

where μ is a parameter and s is an arbitrary complex number. These generalize Carlitz's earlier definitions of $\delta_s^1=\delta_s$ and $\rho_s^1(M, 1)=\rho_s(M)$ [Trans. Amer. Math. Soc. **35**, 397–410 (1933); Proc. London Math. Soc. (2) **38**, 116–124 (1934)] and the introduction of the γ's simplifies computations in the proofs of identities like the following. If $\deg M=f=2ek$, $k\geqq 0$, $e>0$, and if α, β and ϵ are nonzero elements of $GF(p^n)$ such that $\alpha+\beta=\epsilon$, then

$$\sum\rho_s^e(A, \mu)\cdot\rho_t^e(B, \nu)=\rho_{s+t+1}^e(F, \mu\nu),$$

where the summation is over all A and B such that $\deg A=\deg B=f$, $\alpha A+\beta B=\epsilon F$. It is shown that the ρ's, with $\mu=1$ and integral values of s, are the numbers of sets of polynomials satisfying certain equations. *R. Hull.*

Brauer, Richard. On Artin's L-series with general group characters. Ann. of Math. (2) 48, 502–514 (1947).

Ce travail donne la démonstration de la célèbre hypothèse d'Artin, formulée il y a une vingtaine d'années, que la série $L(s, \chi, K/F)$, où K est une extension galoisienne d'un corps F de nombres algébriques de degré fini, et où χ est un caractère du groupe de Galois $G_{K/F}$ de K/F, est le produit de puissances entières de séries $L(s, \xi^*, K/F)$, pour les caractères ξ^* de $G_{K/F}$ induits, au sens de Frobenius, par des caractères ξ de degré 1 de sous-groupes de $G_{K/F}$, ce qui revient à démontrer que tout caractère χ de $G_{K/F}$ peut se représenter comme une combinaison linéaire à coefficients entiers de tels caractères.

Soient ζ, $\bar{Z}$, Z les caractères des groupes g, $\bar{G}$, G, $g\subset\bar{G}\subset G$, tels que ζ induit $\bar{Z}$ et $\bar{Z}$ induit Z. Alors ζ induit Z dans G. Ce fait permet à l'auteur de procéder comme suit: il choisit certains sous-groupes G_n de $G_{K/F}$ et il démontre: (1) que tout caractère χ de $G_{K/F}$ est une combinaison linéaire à coefficients entiers rationnels des charactères ψ^* induits par les caractères ψ des G_n; (2) tout caractère de tout G_n y est induit par un caractère de degré 1 d'un sous-groupe convenable de G_n.

La démonstration de (1) se compose d'une partie locale: pour tout idéal premier $\mathfrak{q}$ d'un corps convenable, χ s'exprime comme combinaison linéaire des ψ^* à coefficients $\mathfrak{q}$-entiers, et d'un passage du local au global, dont voici l'idée: exprimons les ψ_j^* $(j=1, \cdots, k)$ comme combinaisons linéaires des

caractères irreductibles φ_i $(i=1, \cdots, h)$ de $G_{K/F}$, soient $\psi_j^* = \sum \gamma_{ji} \varphi_i$, à coefficients rationnels. Le théorème local montre que le p.g.c.d. des mineurs d'ordre h de la matrice $\Gamma = (\gamma_{ji})$ est 1, et, en soumettant les ψ_j^* à une transformation unitaire à coefficients entiers rationnels C convenable, on aura $C(\psi_j^*) = C\Gamma(\varphi_i)$, où $C\Gamma$ a un mineur d'ordre h égal à ± 1, ce qui suffit.

Soient $\mathfrak{K}_1, \cdots, \mathfrak{K}_h$ les classes d'éléments conjugués de $G_{K/F}$; un caractère χ de $G_{K/F}$ peut s'identifier avec le vecteur $(\chi(\mathfrak{K}_1), \cdots, \chi(\mathfrak{K}_h))$. L'auteur remarque que χ est une combinaison linéaire des ψ^* à coefficients $\mathfrak{q}$-entiers si l'on peut choisir un ensemble $\Psi_{\mathfrak{q}}^*$ de ces caractères tel que toute congruence $\sum_l c_l \psi^*(\mathfrak{K}_l) \equiv 0 \pmod{\mathfrak{q}^t}$ à coefficients $\mathfrak{q}$-entiers c_l qui est vraie pour tout $\psi^* \in \Psi_{\mathfrak{q}}^*$ est aussi vraie pour χ. Pour démontrer le théorème local pour $\mathfrak{q}$, il suffit d'établir ce fait pour les φ_i $(i=1, \cdots, h)$. Pour tout élément A de $G_{K/F}$ d'ordre premier à q, choisissons un q-groupe de Sylow (où q est le premier rationnel que divise $\mathfrak{q}$) $Q(A)$ du normalisateur $\mathfrak{N}(A)$ de A, et soit $S(A) = AQ(A)$. En vertu du théorème de Sylow, les $S(A)$ obtenus par un autre choix de $Q(A)$ sont conjugués du précédent. D'autre part, puisque tout élément de $G_{K/F}$ est produit de deux éléments permutables et univoquement déterminés, d'ordres puissance de q et premier à q, tout $\mathfrak{K}_l$ $(l=1, \cdots, h)$ est non-disjoint avec un et un seul des $S(A)$ choisis.

Soit $\mathfrak{H}(A)$ le groupe engendré par $S(A)$, donc $\mathfrak{H}(A) = (A) \times Q(A)$, où (A) est le groupe cyclique engendré par A. L'auteur définit certaines combinaisons linéaires ψ_j des caractères de $\mathfrak{H}(A)$ qui sont nulles pour tout $\mathfrak{K}_l$ $(l=1, \cdots, h)$ disjoint avec $S(A)$. Les ψ_j^* correspondants sont des combinaisons linéaires des φ_i, mais, inversement, en vertu du théorème de dualité de Frobenius, les restrictions des φ_i à $S(A)$ y sont des combinaisons linéaires de ces ψ_j, ou, ce qui revient au même dans ce cas, des caractères ϑ de $Q(A)$, et le théorème de Frobenius donne une relation entre ces deux transformations. L'examen des matrices correspondantes permet à l'auteur de démontrer à leur propos, par une méthode analogue à celle de son travail antérieur [Ann. of Math. (2) **42**, 53–61 (1941); ces Rev. **2**, 215], un lemme arithmétique, point central de sa démonstration, d'où, par des calculs matriciels, on déduit que si, pour chacun de ces ψ_j, on a $\sum c_l \psi_j^*(\mathfrak{K}_l) \equiv 0 \pmod{\mathfrak{q}^t}$, où $\mathfrak{K}_l$ parcourt les classes non-disjointes avec $S(A)$, on a la même congruence pour tout φ_i.

Si l'on prend toutes les expressions ψ^* ainsi définies pour tous les $S(A)$ choisis, une relation $\sum_{l=1}^h c_l \psi^*(\mathfrak{K}_l) \equiv 0 \pmod{\mathfrak{q}^t}$ a comme conséquence, pour tout $S(A)$ choisi et pour tous les ψ correspondants, $\sum' c_l \psi^*(\mathfrak{K}_l) \equiv 0 \pmod{\mathfrak{q}^t}$, où $\mathfrak{K}_l$ parcourt dans $\sum'$ seulement les classes non disjointes avec $S(A)$; donc aussi, pour tout χ, on a $\sum' c_l \chi(\mathfrak{K}_l) \equiv 0 \pmod{\mathfrak{q}^t}$. En additionnant ces congruences pour tous les $S(A)$, on trouve $\sum_{l=1}^h c_l \chi(\mathfrak{K}_l) \equiv 0 \pmod{\mathfrak{q}^t}$, d'où résulte que tout χ est une combinaison linéaire à coefficients $\mathfrak{q}$-entiers des ψ^*.

La partie (2) de demonstration est basée sur ce que tout caractère de $\mathfrak{H}(A) = (A) \times Q(A)$ est produit d'un caractère ξ de (A) par un caractère ϑ du q-groupe $Q(A)$, et que si ϑ est induit par ϑ', $\xi\vartheta$ l'est par $\xi\vartheta'$. Un q-groupe étant nilpotent, l'auteur démontre facilement par induction que tout son caractère est induit par quelque caractère de degré 1, ce qui achève sa demonstration. *M. Krasner* (Paris).

ANALYSIS

Aczél, John. The notion of mean values. Norske Vid. Selsk. Forh., Trondhjem **19**, no. 23, 83–86 (1947).

For a fixed positive integer n, a function $M(x_1, \cdots, x_n)$ is called a mean value function if it is symmetric in its variables and $M(x, \cdots, x) = x$. It is a normal mean value if in addition it is continuous in $(x_1, \cdots, x_n)$ and is a monotone increasing function of each of its variables, and in addition satisfies the bisymmetric condition that

$$M\{M(x_{11}, \cdots, x_{1n}), \cdots, M(x_{n1}, \cdots, x_{nn})\}$$

is symmetric in its n^2 variables. This above definition differs from a previous one in that the condition of bisymmetry is substituted for a recursive formula defining M for any number of variables. With the present definition, the author establishes the theorem of Kolmogoroff and Nagumo: a necessary and sufficient condition that a function $M(x_1, \cdots, x_n)$ is a normal mean value is that it admits an expression of the form

$$(1) \qquad M(x_1, \cdots, x_n) = F^{-1}\left\{\frac{F(x_1) + \cdots + F(x_n)}{n}\right\},$$

where $F(x)$ is continuous and monotone. The function $F(x)$ is called the Kolmogoroff-Nagumo function relative to M. The sufficiency of the condition is immediate. The necessity is established by exhibiting a function F, depending on M, for which (1) is satisfied. *E. F. Beckenbach.*

Aczél, John. A generalization of the notion of convex functions. Norske Vid. Selsk. Forh., Trondhjem **19**, no. 24, 87–90 (1947).

In terms of normal mean values [see the preceding review] $m(x_1, \cdots, x_n)$ and $M(x_1, \cdots, x_n)$, the author gives the following generalization of the notion of convex function: a function $\Phi(x)$ is said to be convex with respect to (m, M) provided $\Phi\{m(x_1, \cdots, x_n)\} \leqq M\{\Phi(x_1), \cdots, \Phi(x_n)\}$. It is shown that $\Phi(x)$ is convex with respect to (m, M) if and only if $\Phi(x) \equiv G^{-1} f F(x)$, where $f(x)$ is a convex function and $F(x)$ and $G(x)$ are the Kolmogoroff-Nagumo functions relative to m and M, respectively. For $\Phi(x) = x$, the theorem reduces to a known result [G. H. Hardy, J. E. Littlewood and G. Pólya, Inequalities, Cambridge University Press, 1934, p. 89]. *E. F. Beckenbach* (Los Angeles, Calif.).

Laguardia, Rafael. On the extension of an inequality of Tchebycheff. Bol. Fac. Ingen. Montevideo **3** (Año 10), 229–231 (1946). (Spanish)

Let $f(x)$ and $g(x)$ be monotonic and let $\alpha(x)$ be always between $\alpha(a)$ and $\alpha(b)$. Then

$$\int_a^b f(x)g(x)d\alpha(x) \int_a^b d\alpha(x) \geqq \int_a^b f(x)d\alpha(x) \int_a^b g(x)d\alpha(x)$$

if $f(x)$ and $g(x)$ vary in the same sense, with the inequality reversed if they vary in opposite senses.

R. P. Boas, Jr. (Providence, R. I.).

Calculus

✱de Losada y Puga, Cristóbal. Curso de Análisis Matemático. [Course of Mathematical Analysis]. Vol. 2. Universidad Católica del Peru, Lima, 1947. xxi+701 pp. (Spanish)

The first volume appeared in 1945; cf. these Rev. **6**, 225. This volume contains material customary in an "advanced

calculus": theory of series, including power series; applications of the calculus to curves and surfaces; approximate integration, improper integrals, transformation of multiple integrals. *R. P. Boas, Jr.* (Providence, R. I.).

Giambelli, Giovanni. Formole d'integrazione per le funzioni razionali fratte. Atti Accad. Peloritana. Cl. Sci. Fis. Mat. Biol. (2) **5(42)**, 130–150 (1940).

Levi, Beppo. An exercise on elliptic integrals. Math. Notae **6**, 167–190 (1946). (Spanish)

The author's problem is to find the volume common to an elliptic cylinder and an ellipsoid whose center is on the axis of the cylinder.

✱Laboccetta, Letterio. Generazione geometrica delle funzioni discontinue di variabile continua. Atti Secondo Congresso Un. Mat. Ital., Bologna, 1940, pp. 133–138. Edizioni Cremonense, Rome, 1942.

Considering, as a function of the length of arc described by a point moving on a curve called the trajectory, the angle under which a curve called the base is seen from it, and taking as bases and trajectories the line, the circle, the parabola, we obtain the elementary forms of discontinuous functions. *From the author's summary.*

Germay, R.-H.-J. Sur les fonctions implicites. Bull. Soc. Roy. Sci. Liége **15**, 112–116 (1946).

For $F(x, z)=0$, with F analytic in x and z, and $\partial F/\partial z\neq 0$, a method of successive approximations is used to prove the existence of a solution $z=f(x)$. *P. Franklin.*

Germay, R.-H.-J. Sur le théorème des fonctions implicites et la formule de Lagrange. Bull. Soc. Roy. Sci. Liége **15**, 62–68 (1946).

A proof of Lagrange's expansion for z in terms of the x_i when $f(x_1, \cdots, x_n, z)=0$, where f is analytic in all the variables. *P. Franklin* (Cambridge, Mass.).

Challier, Jean. Extension de la formule de Riemann aux intégrales non lineaires. Ann. Univ. Lyon. Sect. A. (3) **5**, 37–39 (1942).

The paper gives a formula for changing a line integral into a double integral. [The result was announced in C. R. Acad. Sci. Paris **214**, 940–942 (1942); these Rev. **5**, 202. The author has also treated the case of three variables: same Ann. (3) **7**, 46–56 (1944); these Rev. **8**, 16.] *F. G. Dressel* (Durham, N. C.).

Theory of Sets, Theory of Functions of Real Variables

Denjoy, Arnaud. L'ordination des ensembles. C. R. Acad. Sci. Paris **224**, 1081–1083 (1947).

The author considers the controversial propositions (where $\mathfrak{m}$ and $\mathfrak{n}$ are infinite cardinals): (I) $\mathfrak{m}=2\mathfrak{m}$, (II) $\mathfrak{m}=\mathfrak{m}^2$, (III) there is no $\mathfrak{n}$ such that $\mathfrak{m}<\mathfrak{n}<2^{\mathfrak{m}}$, (IV) every class can be well-ordered [Zermelo]. None of these has been established in a way to satisfy all analysts. The author believes that it is not possible to prove that any of these is false and he is prepared to reject concepts or hypotheses which are in contradiction with any consequences of these four propositions. Let E be an infinite class. Let I be the class of all orderings of E. Let H be the class of functions on E assuming one of the values 0, 1. Denote by $\overline{\overline{X}}$ the cardinal of X. From (I) and (II) it follows that $\overline{\overline{I}}=\overline{\overline{H}}$ and the author therefore regards the proposition "It is false that $\overline{\overline{I}}=\overline{\overline{H}}$" as erroneous. From (II), (III) and (IV) it follows that E can be ordered in such a way that the cardinal of the class of its distinct final sections exceeds $\overline{\overline{E}}$ and so the author rejects the contradictory result. *J. Todd* (London).

Denjoy, Arnaud. Les ensembles rangés. C. R. Acad. Sci. Paris **224**, 1129–1132 (1947).

Certain properties of ranked series R are established. These series are generalisations of well-ordered series [cf. the author's L'Énumération Transfinie, I, Gauthier-Villars, Paris, 1946; these Rev. **8**, 254]. For instance, an R cannot be represented in two distinct ways as the sum of initial and final sections C_1, F_1 and C_2, F_2, where C_1 is ordinally similar to C_2 and F_1 to F_2. The sum of a ranked series of ranked series is not necessarily ranked but that of a well-ordered series of ranked series is. The product of two ranked orderings is a ranked ordering. *J. Todd* (London).

Kurepa, G. Ensembles de suites dénombrables d'entiers (contribution au problème de Suslin). Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **11**, 59–74 (1947). (French and Russian)

Let $\mathfrak{C}$ be the class of all series $A=\{a_\xi\}_{\xi<\alpha}$, α being a transfinite number of the second class and a_ξ an integer for each $\xi<\alpha$. This class can be partially ordered by saying that A precedes B if the series A is an initial segment of the series B. The problem, as to whether every more than countable sub-class $\mathfrak{T}$ of $\mathfrak{C}$ satisfies at least one of the two conditions following, is known to be equivalent to the problem of Suslin [cf. the author's thesis, Publ. Math. Univ. Belgrade **4**, 1–138 (1935)]: (1) $\mathfrak{T}$ contains a more than countable ordered subclass, (2) $\mathfrak{T}$ contains a more than countable subclass, no two elements of which are comparable. It is shown that two special types of classes $\mathfrak{T}$ satisfy condition (2). *J. Todd* (London).

Sudan, Gabriel. Sur les singularités des fonctions transfinies. Disquisit. Math. Phys. **1**, 315–320 (1941).

Functions $g(\alpha, \nu)$ [of two variables, each ordinal numbers] whose values are ordinal numbers and which satisfy certain conditions of magnitude, monotony and continuity are considered. An important special case is that when $g(\alpha, \nu)=\alpha+\nu$. Such functions possess singularities in the sense of E. Jacobsthal [Math. Ann. **66**, 145–194 (1909), p. 153], i.e., a number ξ such that $g(\alpha, \xi)=\xi$ for all $\alpha<\xi$. A function of four variables is defined by recurrence in terms of g; the values of this function, when one of its arguments assumes the values ω, $\omega\cdot 2$, $\omega\cdot 3$, $\cdots$, are shown to be the singularities of g, arranged in order of magnitude. *J. Todd.*

Kuratowski, Casimir. Sur l'extension de deux théorèmes topologiques à la théorie des ensembles. Fund. Math. **34**, 34–38 (1947).

Let X be a set of infinite power $\mathfrak{m}$; let ϕ be a class, of power less than or equal to $\mathfrak{m}$, of mappings with domains in X and ranges, of power $\mathfrak{m}$, in X. Then there is a class F, of power $2^{\mathfrak{m}}$, of subsets of X such that $A, B\epsilon F$, $A\neq B$ and $f\epsilon\phi$ implies that $f(A)-B$ is of power $\mathfrak{m}$. One corollary: there exists a family of $2^{\mathfrak{c}}$ sets in the line none of which is a continuous image of the other. *R. Arens.*

Sierpiński, Wacław. L'hypothèse généralisée du continu et l'axiome du choix. Fund. Math. 34, 1–5 (1947).

Denote by $\mathbf{H}$ the hypothesis that, $\mathfrak{m}$ being any cardinal such that $\mathfrak{m} \geqq \aleph_0$, there is no cardinal $\mathfrak{n}$ such that $\mathfrak{m}<\mathfrak{n}<2^{\mathfrak{m}}$. Denote by $\mathbf{H}_{\mathfrak{m}}$ this hypothesis for a particular $\mathfrak{m}$. A. Lindenbaum and A. Tarski [C. R. Soc. Sci. Varsovie. Cl. III. 19, 299–330 (1926), p. 314] stated without proof that $\mathbf{H}$ implies the axiom of choice and that if, for a given $\mathfrak{m}$, $\mathbf{H}_{\mathfrak{p}}$ is true for $\mathfrak{p}=\mathfrak{m}$, $2^{\mathfrak{m}}$ and $2^{2^{\mathfrak{m}}}$, then $\mathfrak{m}$ is an aleph. The author establishes the first result and then, by slight modifications of the proof, obtains the second also. *J. Todd.*

Alexiewicz, Andrzej. On Hausdorff classes. Fund. Math. 34, 61–65 (1947).

C. Arzelà gave a well-known necessary and sufficient condition (quasi-uniform convergence) for the limit function of a sequence of continuous functions to be continuous itself. This was generalized from continuous functions to functions of a Baire class α by B. Gagaeff [Fund. Math. 18, 182–188 (1932)] and by H. Fried [Monatsh. Math. Phys. 38, 301–314 (1931)]. The author further generalizes the results of Gagaeff, giving necessary and sufficient conditions for the limit function of a convergent sequence of functions of a certain class $\mathbf{K}$ to belong to $\mathbf{K}$ also. He discusses several general cases of such classes $\mathbf{K}$ and, in particular, those which he designates by $\mathbf{H}^*$, $\mathbf{H}_1^*$, and $\mathbf{H}_2^*$.

(1) Let X be an arbitrary set and Y a separable metric space; let $\mathbf{H}$ be a σ-ring of subsets of X, containing also the empty set. The family $\mathbf{H}^*$ of all functions $f(x)$ from X to Y is called a "Hausdorff class" by the author if for every open set $G\subset Y$ the set $E_x[f(x)\epsilon G]$ belongs to $\mathbf{H}$. (2) Now let X be a topological space and let $\mathbf{R}$ be a class of subsets of X satisfying the following conditions: every subset of a set $X_1\epsilon\mathbf{R}$ belongs to $\mathbf{R}$; the sum of a sequence of sets of $\mathbf{R}$ belongs to $\mathbf{R}$; no set of $\mathbf{R}$ contains any open set. Then denote by $\mathbf{H}_1$ the family of all sets which can be written in the form $G+R$, where G is open and $R\epsilon\mathbf{R}$. The corresponding Hausdorff class $\mathbf{H}_1^*$ is then identical with the family of all functions from X to Y whose points of discontinuity form a set of $\mathbf{R}$. (3) The class $\mathbf{H}_2^*$ is the family of all approximately continuous functions in a Euclidean space X. *A. Rosenthal* (Lafayette, Ind.).

Buck, Ellen F., and Buck, R. C. A note on finitely-additive measures. Amer. J. Math. 69, 413–420 (1947).

The authors call a finitely additive measure Ω on a Boolean algebra $\mathfrak{M}$ of subsets of a countable set X separable if there exists a countable subclass $\mathfrak{M}_0$ of $\mathfrak{M}$ such that (i) $\mathfrak{M}$ is the finite Carathéodory closure of $\mathfrak{M}_0$ with respect to Ω, (ii) $\mathfrak{M}_0$ contains no infinite sets of measure zero and (iii) $\Omega(A)$ is rational for every A in $\mathfrak{M}_0$. If I is the set of positive integers, let $\mathfrak{D}_0$ be the Boolean algebra generated by finite sets and arithmetic progressions, for B in $\mathfrak{D}_0$ let $\Delta(B)$ be the density of B, and let Δ^* be the extension of Δ to the finite Carathéodory closure $\mathfrak{D}_0^*$ of $\mathfrak{D}$ with respect to Δ. (1) If Ω is a separable measure on $\mathfrak{M}$, there exists a one-to-one map T of X onto I such that $T(\mathfrak{M}_0)\subset\mathfrak{D}_0$, $T(\mathfrak{M})\subset\mathfrak{D}_0^*$, and $\Omega(A)=\Delta^*(T(A))$ for all A in $\mathfrak{M}$. (2) If I is topologized by choosing for neighborhoods all arithmetic progressions then there exists a continuous function f from I into $(0, 1)$ such that, for every open interval (a, b) in $(0, 1)$, $S=f^{-1}((a, b))\epsilon\mathfrak{D}_0^*$ and $\Delta^*(S)=b-a$. Some further results concerning the progression topology of I are derived; for example, the set of squares is closed, perfect and nowhere dense. *P. R. Halmos* (Princeton, N. J.).

Halmos, Paul R. On the set of values of a finite measure. Bull. Amer. Math. Soc. 53, 138–141 (1947).

Let E be a set and let S be a σ-set of subsets of E (that is, S is a class of sets containing E and closed under complementation and the formation of countable unions). Let μ be a finite, nonnegative and totally additive measure defined on S. Concerning the values of such measures the reviewer previously proved the following two theorems. (I) The set of values of μ is closed. (II) If μ and ν are measures in the above sense both defined on the same σ-field S, then the set of points of the form $(\mu(A), \nu(A))$, where $A\epsilon S$, is a closed subset of the plane [Danske Vid. Selsk. Math.-Fys. Medd. 21, no. 9 (1945); these Rev. 7, 279]. The purpose of the present note is to give simpler and more direct proofs of these theorems. This has succeeded for the first theorem. Concerning the proof of the second theorem, however, the reviewer finds that the statement in lemma 5, as well as the proof, is not correct; the statement in this lemma is only valid in certain special cases. The proof of theorem 2 is therefore incomplete. [The author agrees with this comment and will rectify the error in a forthcoming paper.] *K. R. Buch* (Copenhagen).

Ridder, J. Ueber Definitionen von Perron-Integralen. I. Nederl. Akad. Wetensch., Proc. 50, 369–377 = Indagationes Math. 9, 227–235 (1947).

In this paper it is shown in a simple way that the definition of an integral given by McShane ["Integration," Princeton University Press, 1944, p. 313; these Rev. 6, 43] is equivalent to the Perron-Bauer definition $\mathfrak{P}^*$. If for $\epsilon>0$ there exists for a function $f(x)$ an upper adjoined function on the right $\psi_r(x)$, and a lower adjoined function on the left $\varphi_l(x)$, with $\psi_r-\varphi_l<\epsilon$, then $f(x)$ is P_2-integrable. If there exist adjoined functions $\psi_l(x)$, $\varphi_r(x)$ with $\psi_l-\varphi_r<\epsilon$, then $f(x)$ is P_2^*-integrable.

If P_1 is the McShane integral and D^*, D the special and general Denjoy integrals, respectively, DK the Denjoy-Khintchine integral, then the following relations hold: $P_1=\mathfrak{P}^*=D^*\subset P_2\subset D$; $P_1=\mathfrak{P}^*=D^*\subset P_2^*\subset D$. The following relations fail to hold: $DK\subseteq P_2$, $P_2\subseteq DK$; $DK\subseteq P_2^*$, $P_2^*\subseteq DK$; $P_2\subseteq P_2^*$, $P_2^*\subseteq P_2$. *R. L. Jeffrey* (Kingston, Ont.).

Cesari, Lamberto. Sulla rappresentazione delle superficie continue di area finita secondo Lebesgue. Ist. Lombardo Sci. Lett. Cl. Sci. Mat. Nat. Rend. (3) 10(79), 31 pp. (1946).

In his work on the theory of Lebesgue area, in which he recently settled the basic previously unsolved problem of characterizing the surfaces of finite Lebesgue area completely by their projection properties [Ann. Scuola Norm. Super. Pisa (2) 10, 253–295 (1941); 11, 1–42 (1942); 12, 61–84 (1943); Atti Accad. Italia. Mem. Cl. Sci. Fis. Mat. Nat. 14, 891–951 (1944); these Rev. 8, 257, 142] Cesari has now turned to the problem of generalized conformal representation. The possibility of such a representation was first proved by Morrey for a wide class of surfaces [Amer. J. Math. 57, 692–702 (1935)]. In this paper Cesari extends Morrey's result by showing that a generalized conformal representation exists for every surface of the type of the closed disc which has finite Lebesgue area and the additional property that no continuum of constancy of one particular (and hence of any) representation separates the plane. Every such surface has a representation f on the closed disc A with the following properties. Let C be the union of all those continua of constancy of f which meet the boundary of A. Then $B=A-C$ is an open set, f is

absolutely continuous in the sense of Tonelli on B, the classical functions $E-G$ and F vanish almost everywhere on B and the Lebesgue area of the surface equals one half the Dirichlet integral of f over B.

In a recent abstract, the author has announced a similar theorem for arbitrary surfaces of finite Lebesgue area [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 1, 509–514 (1946); these Rev. 8, 258]. *H. Federer.*

Theory of Functions of Complex Variables

Montel, Paul. Sur le rôle des familles de fonctions dans l'analyse moderne. Bull. Soc. Roy. Sci. Liége 15, 262–278 (1946).

Morse, Marston, and Heins, Maurice. Deformation classes of meromorphic functions and their extensions to interior transformations. Acta Math. 79, 51–103 (1947).

Les auteurs se sont proposés deux objectifs. D'une part, utiliser les méthodes et les concepts de la topologie pour étendre certaines propositions de la théorie des fonctions méromorphes aux transformations intérieures au sens de Stoïlow [Leçons sur les Principes Topologiques de la Théorie des Fonctions Analytiques, Gauthier-Villars, Paris, 1938]; d'autre part, distinguer les propriétés qui peuvent être étendues de celles qui ne le peuvent pas. Dans des mémoires antérieurs [voir le compte-rendu suivant] les auteurs avaient déjà établi que des propositions comportant des relations entre le nombre des zéros, des pôles et des images des points de ramification de la fonction inverse situés dans un contour et les propriétés de ce contour restent valables après des homéomorphies arbitraires. Dans cette nouvelle étude, ils montrent qu'il n'y a aucune différence entre transformations méromorphes et transformations intérieures d'un disque $|z|<1$ sur la sphère w de Riemann en ce qui concerne les invariants qui caractérisent les classes de déformation de fonctions admettant des zéros, pôles et antécédents de points de ramification donnés. Au contraire, lorsqu'on considère les suites de fonctions au point de vue de la convergence uniforme et des points de Julia, les propriétés des suites méromorphes ne s'étendent pas.

Dans ce mémoire, les auteurs se bornent au cas où l'ensemble caractéristique constitué par les zéros, les pôles de $w=f(z)$ et les points fournissant les points de ramification de son inverse, situés dans $|z|<1$, est fini et ils supposent en général que ces points sont simples. Les déformations admises de $f(z)$ sont définies par $w=F(z,t)$, $0\leqq t\leqq 1$, la fonction F étant continue et $F(z,0)\equiv f(z)$; pour chaque t, la transformation est intérieure et les points de l'ensemble caractéristique varient continûment, enfin, pour $t=1$, les points de l'ensemble caractéristique coïncident globalement avec les points caractéristiques de même nature de l'ensemble initial. Des déformations de types particuliers sont introduites, notamment les déformations restreintes qui laissent invariant l'ensemble caractéristique, et les déformations finalement restreintes dont l'ensemble caractéristique final coïncide avec l'ensemble initial. Les invariants sont n nombres (n étant le nombre des zéros et des pôles) attachés à la fonction f et à l'ensemble caractéristique correspondant; ils ne changent pas dans les déformations restreintes. Des modèles de transformations intérieures avec un ensemble caractéristique et des invariants donnés a priori sont construits, ainsi que des exemples correspondants de transformations méromorphes. Une condition nécessaire et suffisante pour que deux transformations appartiennent à une même classe de déformation restreinte est que les invariants soient les mêmes; dans le cas des fonctions méromorphes, la déformation peut être réalisée au moyen de fonctions méromorphes. Les autres types de déformation sont aussi étudiés.

La définition des invariants repose sur l'étude topologique des déformations admissibles des arcs continus qui sont caractérisés par un ordre différentiel. Cette notion jointe aux propriétés de l'ordre angulaire défini et étudié par les auteurs dans le second de leurs mémoires cités ci-dessous, permet d'introduire les invariants. *G. Valiron* (Paris).

Morse, Marston, and Heins, Maurice. Topological methods in the theory of functions of a complex variable. Bull. Amer. Math. Soc. 53, 1–14 (1947).

Here the authors present some of the salient topics and ideas developed in detail in a sequence of papers [Morse, Duke Math. J. 13, 21–42 (1946); Morse and Heins, Ann. of Math. (2) 46, 600–624, 625–666 (1945); 47, 233–273 (1946); these Rev. 7, 448; 8, 21, and the paper reviewed above]. Particular emphasis is given to these topics: (1) pseudo-harmonic functions, (2) interior transformations with locally simple boundary images, (3) deformation classes of interior or meromorphic functions. In the case of a pseudo-harmonic function with suitable boundary conditions one has $M-S=2-\nu+s-m$, where M is the number of logarithmic poles, S the number of interior saddle points, ν the number of bounding curves, s and m the number of saddle and minimum points of the boundary. The proof of this result and the generalizations rest upon extensions of the topological critical points developed by Morse and involving the use of singular cycles in place of Vietoris cycles. In the case of an interior transformation $w=f(z)$ one has the significant relation $2n(a)=2-\nu+\mu+2q(a)-p$ in which $n(a)$ is the number of zeros of $f(z)-a$, $q(a)$ the sum of the orders of the images g_i of the bounding curves with respect to $w=a$, p the sum of the angular orders of g_i, μ the sum of the orders of the branch elements of f^{-1}. Again the proof rests heavily upon topological methods. Finally the authors discuss deformation or homotopy classes of meromorphic functions with prescribed characteristic sets (zeros, poles, branch point antecedents). A set of invariants is given characterizing these deformation classes. The similarities and differences between the theory of interior transformations and meromorphic functions are described.

M. R. Hestenes (Los Angeles, Calif.).

Demin, E. Remarque sur une formule de Laurent. Bull. Soc. Roy. Sci. Liége 15, 84–86 (1946).

Laurent's formula is

$$f(z)=f(a_1)+\sum_{n=2}^{\infty}F_{n-1}(z)\left\{\frac{f(a_1)}{F_n'(a_1)}+\cdots+\frac{f(a_n)}{F_n'(a_n)}\right\},$$

where $a_n\to a$, $F_n(z)=(z-a_1)\cdots(z-a_n)$, and $f(z)$ is analytic in a circle about a containing all the a_n. The author transforms this formula by conformal mapping to apply to functions analytic in a strip. *R. P. Boas, Jr.*

Germay, R.-H.-J. Extension du théorème d'E. Picard sur la décomposition en facteurs primaires des fonctions uniformes ayant une ligne de points singuliers essentiels. Bull. Soc. Roy. Sci. Liége 15, 9–13 (1946).

Let $\{a_n\}$ be an infinite sequence whose only limit points are on a given circle, with an infinite number of a_n outside

and inside the circle. The author shows how to construct a function analytic both inside and outside the circle with zeros of arbitrary orders μ_n at a_n. *R. P. Boas, Jr.*

Germay, R.-H.-J. Sur les produits indéfinis de facteurs primaires. Bull. Soc. Roy. Sci. Liége **14**, 476–478 (1945).

The author constructs an entire function having zeros of arbitrary orders μ_n at the points a_n, where $|a_n|\to\infty$.
R. P. Boas, Jr. (Providence, R. I.).

Valiron, Georges. On some points of the theory of analytic functions. Bol. Fac. Ingen. Montevideo **3** (Año 10), 185–206 (1946) = Facultad de Ingeniería Montevideo. Publ. Inst. Mat. Estadística **1**, 137–158 (1947). (Spanish)

Three lectures on entire functions.

Tsuji, Masatsugu. On the domain of existence of an implicit function defined by an integral relation $G(x, y)=0$. Proc. Imp. Acad. Tokyo **19**, 235–240 (1943). [MF 14816]

A refinement is given of Julia's theorem concerning analytic functions defined by relations of the form $G(x, y)=0$, where G is entire in (x, y). *M. Heins.*

Tsuji, Masatsugu. On the cluster set of a meromorphic function. Proc. Imp. Acad. Tokyo **19**, 60–65 (1943). [MF 14794]

Extensions are given of the work of Iversen, Kunugui and Noshiro on the cluster sets of meromorphic functions. The work lies in the direction of introducing statements involving the notion of sets of zero capacity.
M. Heins (Providence, R. I.).

Tsuji, Masatsugu. On the Riemann surface of an inverse function of a meromorphic function in the neighbourhood of a closed set of capacity zero. Proc. Imp. Acad. Tokyo **19**, 257–258 (1943). [MF 14818]

An outline of a proof of a theorem treated in a subsequent paper [cf. the following review]. *M. Heins.*

Tsuji, Masatsugu. Theory of meromorphic functions in a neighbourhood of a closed set of capacity zero. Jap. J. Math. **19**, 139–154 (1944). [MF 15000]

The author sketches a theory of functions single-valued and meromorphic in the neighborhood of a closed set of capacity zero along the lines of Nevanlinna's theory. Related questions were treated earlier by G. af Hällström in his thesis [Acta Acad. Aboensis **12**, no. 8 (1940); these Rev. **2**, 275]. *M. Heins* (Providence, R. I.).

Tsuji, Masatsugu. On conformal mapping of an infinitely multiply connected domain. Proc. Imp. Acad. Tokyo **20**, 3–6 (1944). [MF 14865]

The author announces a series of theorems concerning the conformal mapping of regions of infinite connectivity and related questions. Part of this work is open to the same question as some of the author's other investigations in this direction [Jap. J. Math. **18**, 759–775, 977–984 (1943); **19**, 155–188 (1944); these Rev. **7**, 516]. *M. Heins.*

Krzywoblocki, M. Z. A local maximum property of the fourth coefficient of schlicht functions. Duke Math. J. **14**, 109–128 (1947).

The author studies Löwner's representation for the coefficients of functions $f(z)=z+b_2z^2+b_3z^3+\cdots$ which are regular and schlicht in $|z|<1$. The coefficients given by Löwner's formulas lie everywhere dense in the coefficient space of schlicht functions. The representation is in terms of integrals involving Löwner's k-function which has the form $k(t)=e^{i\theta(t)}$, $\theta(t)$ real, $0\leqq t<\infty$. If $\theta(t)$ is constant, the corresponding coefficients b_n belong to the function $f(z)=z/(1+\eta z)^2$, $|\eta|=1$, and so $|b_n|=n$ $(n=2, 3, \cdots)$. The author proves the following result. If $k(t)=e^{i[\alpha_0+\alpha(t)]}$, where α_0 is an arbitrary real constant and $\alpha(t)$ is a real function satisfying $|\alpha(t)|\leqq 1/18000$, then for the corresponding fourth coefficient b_4 we have $|b_4|\leqq 4$. In this sense $|b_4|$ has a local maximum for the function $f(z)=z/(1+\eta z)^2$, $|\eta|=1$.

The representation for b_4 is dissected into a sum of the form $4-J_2-J_4-J_6$ and it is shown that

$$J_2\geqq c_2\int_0^1 X(u)^2du,\quad |J_4|\leqq c_4\int_0^1 X(u)^4du,\quad J_6\geqq 0,$$

where c_2 and c_4 are positive constants and $X(u)=\frac{1}{2}\sin\theta(u)$, $\theta(e^{-t})=\alpha(t)$. *D. C. Spencer* (Stanford University, Calif.).

Nehari, Zeev. Une inégalité dans la théorie des fonctions bornées dans un anneau. C. R. Acad. Sci. Paris **224**, 1093–1095 (1947).

The following theorem is established. Let $f(z)$ be analytic, single-valued and of modulus not exceeding one for $\rho^{\frac{1}{2}}<|z|<\rho^{-\frac{1}{2}}$ $(0<\rho<1)$. If $f'(1)=f'(-1)=0$, then

$$\frac{|f'(z)|}{1-|f(z)|^2}\leqq\frac{|F'(z)|^2}{1-|F(z)|^2},$$

where $F(z)\equiv k^{\frac{1}{2}}\,\mathrm{sn}\,\{-i\theta_3^2(\log iz)\}$ and $\mathrm{sn}\,u$ has the periods $4k=2\pi\theta_3^2(\rho^2)$ and $2ik'=-2i\theta_3^2(\rho^2)\log\rho$. The inequality is sharp. The proof is based on methods due to Ahlfors [Trans. Amer. Math. Soc. **43**, 359–364 (1938)]. Generalizations are given. *M. Heins* (Providence, R. I.).

Nehari, Zeev. Sur les fonctions bornées dans un anneau. C. R. Acad. Sci. Paris **224**, 1135–1137 (1947).

It is shown that, if $f(z)$ is analytic, single-valued, and of modulus not exceeding unity for $\rho^{\frac{1}{2}}<|z|<\rho^{-\frac{1}{2}}$ $(0<\rho<1)$, and if $f(1)=f(-1)$, then

$$|f'(\pm 1)|\leqq 2\rho^{\frac{1}{2}}\prod_1^\infty(1-\rho^{2k})^2\prod_1^\infty(1+\rho^{2k})^4.$$

The proof is based upon a conformal mapping of the annulus, slit along the rectilinear segment with endpoints $-\rho^{-\frac{1}{2}}$ and $-\rho^{\frac{1}{2}}$, onto the interior of the unit circle. *M. Heins.*

Mandelbrojt, S., and MacLane, G. R. On functions holomorphic in a strip region, and an extension of Watson's problem. Trans. Amer. Math. Soc. **61**, 454–467 (1947).

This paper contains the detailed proofs of results announced earlier [C. R. Acad. Sci. Paris **223**, 186–188 (1946); these Rev. **8**, 20]. *R. P. Boas, Jr.* (Providence, R. I.).

Ferrand, Jacqueline. Note on a paper by Mandelbrojt and MacLane. Trans. Amer. Math. Soc. **61**, 468 (1947).

The author points out that the results of the paper reviewed above can be extended to unsymmetrical domains by the use of results of Ferrand and Dufresnoy [Dufresnoy, C. R. Acad. Sci. Paris **220**, 189–190 (1945); Ferrand, ibid., 873–874 (1945); Ferrand and Dufresnoy, Bull. Sci. Math. (2) **69**, 165–174 (1945); these Rev. **7**, 55, 201; **8**, 145].
R. P. Boas, Jr. (Providence, R. I.).

Carathéodory, C. Zum Schwarzschen Spiegelungsprinzip. Comment. Math. Helv. **19**, 263–278 (1946).

This paper is concerned with a group of theorems which are related to the Schwarz reflection principle. Let $f(z)$ denote an arbitrary complex-valued function defined in a

region G of the z-plane and let ζ denote a point of the boundary of G. Then a complex number α (finite or infinite) is said to be a boundary value (Randwert) of f at ζ provided that there exists a sequence of points in G, say $\{z_k\}$, with $\lim_{k\to\infty} z_k=\zeta$, $\lim_{k\to\infty} f(z_k)=\alpha$. The hypothesis of the first principal theorem of the paper is as follows. Let $f(z)$ be meromorphic in $|z|<1$ and let AB denote an arc of $|z|=1$. It is assumed that for each ζ of AB, save those belonging to a fixed set of linear measure zero, there exists a sequence of points of $|z|<1$, say $\{z_k\}$, which lie between two chords of the unit circle meeting in ζ, tend to ζ, and in addition satisfy (1) $\lim f(z_k)$ exists, (2) $\lim f(z_k)$ is real or ∞. Under this hypothesis it is concluded that for each point ζ_0 of AB only the three following possibilities can occur: (a) the set W of all boundary values of $f(z)$ at ζ_0 is the extended plane; (b) W coincides with either the closed upper half plane or the closed lower half plane and contains no inner points of the other half plane; (c) either f or $1/f$ is regular at ζ_0 and the reflection principle prevails in a neighborhood of ζ_0. The principal tool in the proof is Fatou's theorem. Examples of functions satisfying the hypothesis given above and realizing (a) or (b) of the conclusion for all points of $|z|=1$ are given with the aid of the elliptic modular function and related functions.

The author also establishes by related methods and the notion of a normal covering sequence the following theorem. Let $f(z)$ be meromorphic in $|z|<1$ and let ζ_0 denote an arbitrary point of $|z|=1$. Then the Riemann sphere admits a decomposition $H+G_1+G_2+\cdots$ as a sum of mutually disjoint sets, where H is closed and not vacuous and the G_i, if not empty, are regions; this decomposition is defined with the aid of those radial limits of f which exist and is such that for each G_i either every one of its points is a boundary value of f at ζ_0 or none is. Furthermore, every point of H is a boundary value of f at ζ_0. *M. Heins.*

***Nevanlinna, Rolf. Eindeutigkeitsfragen in der Theorie der konformen Abbildung.** C. R. Dixième Congrès Math. Scandinaves 1946, pp. 225–240. Jul. Gjellerups Forlag, Copenhagen, 1947.

In this address the author gives an account of recent investigations concerned with uniqueness problems in the theory of the conformal mapping of abstract Riemann surfaces. The notion of surfaces of closed character is discussed. An abstract Riemann surface F is said to be of closed character if it admits a (1, 1) directly conformal mapping into a closed Riemann surface F_0 and if further this mapping is determined up to a (1, 1) directly conformal mapping of F_0 onto itself. Investigations concerned with this concept are due to Courant, Koebe and more recently to Sario. A summary is given of the author's theory of Abelian differentials on open surfaces [Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. 1 (1941); these Rev. 7, 427]. In connection with this work the author introduces the concept of removable boundary: an abstract open Riemann surface is said to have a removable boundary if every function single-valued and analytic on the surface with finite Dirichlet integral reduces to a constant. This class of surfaces contains those with null boundary. Two sufficient conditions that an open Riemann surface have a removable boundary are given: one in terms of regular parametric representations of an open Riemann surface, the other in terms of a decomposition of a Riemann surface into elementary components and a conformal invariant, the "width measure" (Breitenmass), associated with each of these components.

M. Heins (Providence, R. I.).

***Myrberg, P. J. Über analytische Funktionen auf transzendenten Riemannschen Flächen.** C. R. Dixième Congrès Math. Scandinaves 1946, pp. 77–96. Jul. Gjellerups Forlag, Copenhagen, 1947.

An account is given of the author's recent investigations concerned with the extension of the notion of Abelian integrals to transcendental Riemann surfaces [Acta Math. **76**, 185–224 (1945); Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. nos. **14** (1943); **31** (1945); these Rev. **7**, 57, 428]. The author calls attention to unsolved problems in the field.

M. Heins (Providence, R. I.).

Ibraguimoff, I. I. Sur les systèmes complets de fonctions analytiques. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **11**, 75–100 (1947). (Russian. French summary)

The author proves seven theorems; the first three, dealing with periodic functions, were announced in C. R. (Doklady) Acad. Sci. URSS (N.S.) **52**, 389–390 (1946); these Rev. **8**, 144. The others are as follows. (4) If $f(z)$ is analytic in $|z|<1$ and $F(0)=F'(0)=1$, then $\{F(z^n)\}$ is complete in $|z|<1$. (5) If $F(z)$ is analytic in $|z|\leqq e^{-\delta}$ and $\alpha_k>0$, $\alpha_k\to\infty$, $\sum\alpha_k^{-\beta}=\infty$, where $\beta=\frac{1}{2}\pi/\tan^{-1}(\delta/\pi)$, then $\{F(z^{\alpha_k})\}$ is complete in the circle $|z|<e^{-\delta}$ cut along the negative real axis. (6) If $\lambda_k>0$ and $\sum 1/\lambda_k<\infty$, $F(z)=\sum_{k=1}^{\infty}a_k z^{\lambda_k}$, $|z|\leqq 1$, then $\{F(x^{\alpha_k})\}$, $\alpha_k>0$, is complete in $L^2(0, 1)$ if and only if $\sum 1/\alpha_k=\infty$. (7) If $F(z)$ is analytic in $|z|<e^{\pi+\delta}$ and $|\alpha_n|\to 0$, then $F(z^{\alpha_k})$ is complete in the circle $|z|<e^{-\delta}$ cut along a radius. [It seems to the reviewer that the proof of the sufficiency part of (6) is incomplete.] *R. P. Boas, Jr.*

Bernstein, S. N. Sur la meilleure approximation sur tout l'axe réel par des fonctions entières de degré fini. V. C. R. (Doklady) Acad. Sci. URSS (N.S.) **54**, 475–478 (1946).

[For part IV see the same C. R. (N.S.) **54**, 103–108 (1946); these Rev. **8**, 373.] The author continues his investigation into $A_p f$, the best approximation of $f(x)$ by "functions of degree p," i.e., by entire functions of exponential type p, in $-\infty<x<\infty$, and $E_n(f;\lambda)$, the best approximation of $f(x)$ by polynomials of degree n in $-\lambda\leqq x<\lambda$, comparing A_p and E_n. Functions $f(x)$ are dealt with such that (i) $|f(x)|<H(x)$, $-\infty<x<\infty$, where $H(z)$ is an even entire function of genus zero, with $H(0)>0$, $H^{(k)}(0)\geqq 0$ $(k=1, 2, \cdots)$. The following theorems are proved. (6) If $F_p(z)$ is a function of degree p and satisfies (i), then there are polynomials $P_n(z)$ of degree n such that, whenever $0<c<1$ and $n\geqq N=N(c)$, $|F_p(x)-P_n(x)|\leqq\exp(-\frac{1}{3}nc^{\frac{3}{2}})$ for $|x|\leqq np^{-1}(1-c)$. (7) When $f(x)$ satisfies condition (i) then $\lim_{n\to\infty} E_n(f; n/(p+\epsilon))\to A_p f$, $A_{p-\epsilon}f\to\lim_{n\to\infty} E_n(f; n/p)$ as $\epsilon\to 0$ $(\epsilon>0)$. In the proof of theorem 6 Chebyshev polynomials are employed. Theorem 7 is deduced from theorem 6; it is used to complete a former result of the author [theorem 5; cf. part IV], and to generalise a result due to M. Krein: if $F_p(z)$ is a function of degree p and satisfies (i), and if $F_p(x)>0$, then $F_p(z)=s^2(z)+t^2(z)$ where $s(z)+it(z)$ is of degree $\frac{1}{2}p$ and has no zero for $y>0$ $(z=x+iy)$.

H. Kober (Birmingham).

Ganapathy Iyer, V. On the translation numbers of integral functions. J. Indian Math. Soc. (N.S.) 10, 17–28 (1946).

Let $f(z)$ be an entire function with at least one zero. The complex number λ is said to be a translation number of $f(z)$ if $f(z+\lambda)/f(z)$ is also an entire function. A translation

number λ is of the first kind if $-\lambda$ is also a translation number; all others are said to be of the second kind. The former share some of the properties of periods of analytic functions. If $f(z)$ has two translation numbers of the first kind with nonreal ratio, then there exist two translation numbers λ_1, λ_2, also of the first kind, such that every translation number of $f(z)$ is of the form $m_1\lambda_1+m_2\lambda_2$, where m_1 and m_2 are integers. Thus, $f(z)$ cannot have three independent translation numbers, two of which are of the first kind. If λ is a translation number of the first kind, then $f(z)=\theta(z)\exp h(z)$, where $h(z)$ and $\theta(z)$ are entire functions and $\theta(z)$ has period λ. In contrast, $f(z)$ can have a countable number of independent translation numbers of the second kind. The second section of the paper deals with the solution of the functional equation $f(z+\lambda)=f(z)g(z)$ for given λ and $g(z)$. Necessary and sufficient conditions for the existence of solutions $f(z)$, entire and not identically zero, are given in terms of the zeros of $g(z)$. Some remarks are made concerning the relation of the orders of $g(z)$ and $f(z)$.

R. C. Buck (Providence, R. I.).

Lozinski, S. A generalization of a theorem of S. Bernstein. C. R. (Doklady) Acad. Sci. URSS (N.S.) **55**, 9–12 (1947).

The author considers an entire function $f(z_1, \cdots, z_n)$ of n complex variables satisfying the inequality

$$|f(x_1+iy_1, \cdots, x_n+iy_n)| \leqq C \exp R(y_1^2+\cdots+y_n^2)^{\frac{1}{2}}.$$

The classical theorem of Bernstein [C. R. Acad. Sci. Paris **176**, 1603–1605 (1923)] is generalized to

$$\mathfrak{D}_G\left[\frac{D_l f(x)\cos\alpha+Rf(x)\sin\alpha}{R}\right]\leqq\mathfrak{D}_G[f(x)],$$

where $f(x)=f(x_1, \cdots, x_n)$, $D_l f(x)$ denotes the derivative of $f(x)$ in the direction $l=\{l_1, \cdots, l_n\}$ and $\mathfrak{D}_G[f(x)]$ is any one of a number of integral averages of $f(x)$. The author then considers exponential integrals of the form

$$f(x)=\int\cdots\int e^{i(t_1x_1+\cdots+t_nx_n)}d_tF(E),$$

where the integration is over $\sum t_j^2\leqq R^2$. The conjugate of this function, associated with the direction l, is defined by

$$\tilde{f}_l(x)=i\int\cdots\int e^{i(t_1x_1+\cdots+t_nx_n)}\operatorname{sgn}(t_1l_1+\cdots+t_nl_n)d_tF(E).$$

Several inequalities relating averages of $f(x)$ and of linear combinations of $D_l f(x)$ and $D_l\tilde{f}_l(x)$ are stated. These generalize inequalities proved for the one variable case by Szegö and by Boas [Trans. Amer. Math. Soc. **40**, 287–308 (1936)].

R. C. Buck (Providence, R. I.).

Roure, Henri. Généralisation des fonctions zétafuchsiennes. C. R. Acad. Sci. Paris **224**, 1687–1689 (1947).

Continuation of the author's papers in Ann. Fac. Sci. Univ. Toulouse (4) **6**, 15–31 (1943); **7**, 99–122 (1945); these Rev. **7**, 380.

Bers, Lipman, and Gelbart, Abe. On generalized Laplace transformations. Ann. of Math. (2) **48**, 342–357 (1947).

This paper is an extension and application of the theory of Σ-monogenic functions developed by the same authors in previous papers [Quart. Appl. Math. **1**, 168–188 (1943); Trans. Amer. Math. Soc. **56**, 67–93 (1944); these Rev. **5**, 25; **6**, 86]. A Σ-monogenic function is a function $f(x+iy)=u(x, y)+iv(x, y)$, where $u_x=\tau_1(y)v_y$, $u_y=-\tau_2(y)v_x$. A Σ-exponential function is defined as a power series in "formal" powers. Generalized trigonometric functions corresponding to the ordinary sines and cosines are given and some of their properties are established. A generalized Laplace transformation is defined in which the Σ-exponential function takes the place of the ordinary exponential function in the ordinary Laplace transformation. Regions of convergence of the generalized Laplace transformation are established and an analytic continuation is given for Σ-monogenic functions which admit a representation by means of generalized Laplace integrals.

H. P. Thielman (Ames, Iowa).

Terracini, Alejandro. On the geometry of monodiffric polynomials. Revista Unión Mat. Argentina **12**, 55–61 (1946). (Spanish)

R. P. Isaacs [Univ. Nac. Tucumán. Revista A. **2**, 177–201 (1941); these Rev. **3**, 298] studied the class of monodiffric functions $f(z)$ of a complex variable z. This class is defined by the equality of the two difference quotients $f(z+1)-f(z)=-i\{f(z+i)-f(z)\}=\Delta f(z)$. In particular, the monodiffric polynomials of degree n are defined by $a_0z^{(n)}+\cdots+a_n$, where the monodiffric powers $z^{(n)}$ are obtained by the recursion formula: $\Delta z^{(n)}=nz^{(n-1)}$, $z^{(0)}=1$, $0^{(n)}=1$. The number of roots of a monodiffric polynomial of degree n is at least n and does not exceed n^2.

The author studies the geometry of the roots of monodiffric equations [cf. his paper, Actas Acad. Ci. Lima **8**, 217–250 (1945); these Rev. **8**, 146]. The main case under consideration is that where the roots of a quadratic equation are four in number and all distinct. The four roots are represented geometrically by an orthogonal quadrangle $ABCD$ (the opposite sides are perpendicular) which may be determined as the vertices and orthocenter of a triangle. Necessary and sufficient conditions are obtained such that the vertices of an orthogonal quadrangle represent the roots of a quadratic monodiffric equation.

Also studied are linear combinations with constant coefficients of two quadratic monodiffric equations. In the final part of the paper the author studies the binomial monodiffric equations $z^{(n)}=M$.

J. De Cicco.

Theory of Series

Tenca, Luigi. Progressioni aritmetiche contenute in una progressione aritmetica d'ordine e classe superiori. Period Mat. (4) **24**, 162–167 (1946).

Agnew, Ralph Palmer. A simple and natural notation for the theory of summability of series and sequences. Univ. Nac. Tucumán. Revista A. **5**, 195–202 (1946).

The author proposes the notations $A\{s_n\}$ or $A\{s_0, s_1, \cdots\}$ and $A\{\sum_0^\infty u_n\}$ or $A\{u_0+u_1+\cdots\}$ for the values assigned to the sequence $\{s_n\}$ and to the series $\sum_0^\infty u_n$ by a general summability operator A.

R. C. Buck.

Darevsky, V. On Toeplitz's methods. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **11**, 3–32 (1947). (Russian. English summary)

[In the title of the summary the author's initial was misprinted as Y. and Toeplitz was misprinted as Toepltz.] Let A denote a transformation $y_n=\sum_{k=1}^\infty a_{nk}x_k$ by which a sequence x_n is evaluable to L if $y_n\to L$. Many questions involving relative strength, inclusion and consistency can be answered in terms of inverse matrices when the matrices

are such that $a_{nn} \neq 0$ and $a_{nk}=0$ when $k>n$. Such questions were answered for more general transformations by Mazur [Math. Z. 28, 599–611 (1928)], Banach [Théorie des Opérations Linéaires, Warsaw, 1932] and others. Considering only bounded sequences, this paper extends these studies. The main theorem gives involved sufficient conditions that A and B be consistent over the set of bounded sequences. The conditions are used to construct two methods A and B such that (1) both A and B are regular; (2) some bounded divergent sequences evaluable A are nonevaluable B, and conversely; (3) some bounded divergent sequences are both evaluable A and evaluable B; (4) A and B are consistent over the set of bounded sequences. *R. P. Agnew.*

Teghem, Jean. Sur des procédés de sommation de séries divergentes. Bull. Soc. Roy. Sci. Liége 14, 366–376 (1945).

The author announces results dealing with the general Euler series-to-series transform. Even when it is not regular, this summability method is consistent with convergence, and may sum certain divergent series. He discusses its behavior with respect to omission and insertion of terms of a series. [These results overlap with those of Agnew, Amer. J. Math. 66, 313–338 (1944); these Rev. 6, 46.] The author makes application of the transform to analytic continuation and the solution of certain differential equations of infinite order. *R. C. Buck* (Providence, R. I.).

Avakumović, Vojislav G. Über die Konvergenzbedingung der Inversionssätze der Laplaceschen Transformation. Bull. Intern. Acad. Croate. Cl. Sci. Math. Nat. 34, 49–57 (1941).

The function $\rho(u)$, defined for $u>0$, belongs to the class R-O if for some fixed $\epsilon>0$ and $\lambda>1$ it satisfies

$$(1)\qquad \rho(u)\geqq\epsilon,\quad \rho(u')/\rho(u)=O(1),\qquad 0<u\leqq u'\leqq\lambda u.$$

With this definition, the author proves the following Tauberian theorem. (1) Let $\rho(u)$ and $g(u)$ belong to R-O and suppose that $J(s)=s\int_0^\infty e^{-su}A(u)du$ converges for $s>0$. Suppose that

$$s\int_0^\infty e^{-su}A(u)du=O\{g(1/s)\},\qquad s\to 0,$$

$$\frac{\rho(u')A(u')-\rho(u)A(u)}{\rho(u)g(u)}>-\omega(\lambda),\qquad 0<u<u'\leqq\lambda u,$$

where $\lambda>1$ and $\omega(\lambda)$ depends only on λ. Then $A(u)=O\{g(u)\}$ as $u\to\infty$. (2) Let $\lim_{u\to\infty} L(ux)/L(u)=1$ for every $x>0$ and let $\rho(u)$ be a function of R-O for which

$$\limsup_{u\to\infty}|1-\rho(u')/\rho(u)|=h(\lambda),\qquad 0<u\leqq u'\leqq\lambda u;\ \lambda>1,$$

and $\lim_{\lambda\to 1} h(\lambda)=0$. Suppose that $J(s)=s\int_0^\infty e^{-su}A(u)du$ converges for $s>0$ and that $J(s)\sim s^{-\alpha-1}L(s^{-1})$. Then if

$$\liminf_{u\to\infty}\min_{u\leqq u'\leqq\lambda u}\frac{\rho(u')A(u')-\rho(u)A(u)}{\rho(u)u^\alpha L(u)}=-\omega(\lambda)$$

for every $\lambda>1$ and $\lim_{\lambda\to 1}\omega(\lambda)=0$, we have the conclusion $A(u)\sim u^\alpha L(u)/\Gamma(\alpha+1)$ as $u\to\infty$. *H. R. Pitt* (Belfast).

Avakumović, Vojislav G. On the convergence condition of the O-inversion theorems for the Laplace transformation. Rad Hrvatske Akademije Znanosti i Umjetnosti. Razred Mat.-Prirodoslov. 84, 143–156 (1941). (Croatian)

A summary is reviewed above.

Fourier Series and Generalizations, Integral Transforms

Herrera, Felix E. On the problem of the determination of the jump of functions. Univ. Nac. Tucumán. Revista A. 5, 255–288 (1946). (Spanish)

Let $f(x)$ be an integrable function of period 2π and let D_x be its generalized jump at the point x, as defined by Szász [Duke Math. J. 4, 401–407 (1938)]. The author extends various known results on determining D_x by means of the Fourier series of $f(x)$ to this more general definition. His principal results are

$$(1+\alpha)\pi\lim_{n\to\infty} n^{-1}\{\sigma_n^\alpha(x)\}'=D_x,$$

$$\pi\lim_{n\to\infty}\{\bar\sigma_{\mu_n}^\alpha(x)-\bar\sigma_{\nu_n}^\alpha(x)\}=D_x\log\theta,\qquad \mu_n/\nu_n\to\theta>0,$$

where σ_n^α $(\alpha>0)$ is the nth (C,α) partial sum of the Fourier series and $\bar\sigma_n^\alpha$ is its conjugate. As the author notes in an addendum, similar results were given by Maruyama [Tohôku Math. J. 46, 68–74 (1939); these Rev. 1, 138] and Chow [J. London Math. Soc. 16, 23–27 (1941); 17, 177–180 (1942); these Rev. 3, 105; 4, 244]. The analogous results are given for Fourier integrals. *R. P. Boas, Jr.*

Galbraith, A. S., and Green, J. W. A note on the mean value of the Poisson kernel. Bull. Amer. Math. Soc. 53, 314–320 (1947).

The authors evaluate the integral

$$(2\pi)^{-1}\int_0^{2\pi}\{P(r,\theta)\}^\sigma d\theta,\qquad \sigma>0,$$

where $P(r,\theta)=(1-r^2)/(1-2r\cos\theta+r^2)$, and investigate its order of magnitude as $r\to 1-0$. *L. S. Bosanquet.*

Korous, Josef. On a generalization of Fourier series. Časopis Pěst. Mat. Fys. 71, 1–15 (1946). (English. Czech summary)

Let $f(x)$ be integrable on $[\alpha,\alpha+\pi]$. Let $\{l_\nu\}_{-\infty}^{\infty}$ be a sequence of real numbers such that $l_\nu<l_{\nu+1}$, $l_{-1}<0\leqq l_0$, $l_\nu=\nu+a+\lambda_\nu$, $\limsup|\lambda_\nu|<\frac{1}{16}$. The series considered have the form

$$(1)\qquad \lim_{n\to\infty}\sum_{\nu=-n}^{n}(a_\nu\cos l_\nu x+b_\nu\sin l_\nu x),$$

where

$$\begin{Bmatrix}a_\nu\\ b_\nu\end{Bmatrix}=\frac{k(l_\nu)}{l'(l_\nu)}\int_\alpha^{\alpha+\pi}f(t)\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}l_\nu t\,dt,$$

$$l(z)=(z-l_0)\prod_{\nu=-\infty}^{\infty}{}'(1-z/l_\nu)e^{z/l_\nu},$$

$$k(z)=l(z)\cot\pi z+\pi^{-1}z\sum_{\nu=-\infty}^{\infty}{}'\frac{l(\nu)}{\nu(\nu-z)}-\frac{l(0)}{\pi z}+b;$$

b is an arbitrary real constant. If $f(z)$ is of bounded variation, (1) is equiconvergent with the Fourier series of $f(x)$, uniformly in $(\alpha+\epsilon,\alpha+\pi-\epsilon)$. Under more stringent hypotheses on the l_ν, the equiconvergence holds in larger intervals, and $\lim a_\nu=\lim b_\nu=0$.

This series should be compared with the "nonharmonic Fourier series" of Wiener and Levinson [see Levinson, Gap and Density Theorems, Amer. Math. Soc. Colloquium Publ., v. 26, New York, 1940; these Rev. 2, 180]. In the nonharmonic Fourier series the coefficients are determined differently, the sequence $\{l_\nu\}$ is less restricted and the series

is equiconvergent with the Fourier series of $f(x)$ in an interval of length 2π. *R. P. Boas, Jr.* (Providence, R. I.).

✱Bohr, Harald. Almost Periodic Functions. Chelsea Publishing Company, New York, N. Y., 1947. ii+114 pp. $2.00.

This is a translation by Harvey Cohn (appendices translated by F. Steinhardt) of Ergebnisse der Mathematik, vol. 1, no. 5, Springer, Berlin, 1932.

Bogolioùboff, N. Sur quelques propriétés arithmétiques des presque-périodes. Ann. Chaire Phys. Math. Kiev **4**, 185–205 (1939). (Ukrainian and French)

This paper provides the first direct proof of Bohr's approximation theorem without the use of Fourier series. It was known that in order to give such a proof it would be sufficient to prove that if $f(t)$ is almost periodic then there exists to every $\epsilon>0$ a set of real numbers $\lambda_1, \cdots, \lambda_n$ and a $\delta>0$ such that any τ satisfying the inequalities $|\lambda_1\tau|<\delta, \cdots, |\lambda_n\tau|<\delta$, mod 2π, is a translation number of $f(t)$ belonging to ϵ. With slight simplifications the proof is as follows.

Lemma I. If $n_1, \cdots, n_m$ are different integers in $-M<n<M$ and $M>q\geqq(8M/m)^2$ then there exist q numbers $\lambda_1, \cdots, \lambda_q$ in $0\leqq\lambda<1$ such that any integer n in $-4M<n<4M$ for which $\cos 2\pi\lambda_j n>0$, $j=1, \cdots, q$, is of the form $n=n_p+n_r-n_s-n_t$. Proof. The periodic function $f(n)$ of the integral variable n with period $8M$ which in $-4M\leqq n<4M$ is 1 for $n=n_1, \cdots, n_m$ and 0 for $n\neq n_1, \cdots, n_m$ has an expansion $f(n)=\sum_{k=0}^{8M-1}\alpha_k e^{2\pi ikn/8M}$, where $\alpha_0=m/8M$, $\sum_{k=0}^{8M-1}|\alpha_k|^2=m/8M$ and all $|\alpha_k|\leqq\alpha_0$. Form the convolutions

$$g(n)=(1/8M)\sum_{l=0}^{8M-1} f(n+l)f(l)=\sum_{k=0}^{8M-1}|\alpha_k|^2e^{2\pi ikn/8M},$$

$$h(n)=(1/8M)\sum_{l=0}^{8M-1} g(n+l)g(l)=\sum_{k=0}^{8M-1}|\alpha_k|^4e^{2\pi ikn/8M}.$$

Plainly $h(n)>0$ for an n in $-4M<n<4M$ if and only if $n=(\text{some})\ n_p+n_r-n_s-n_t$. Let $\alpha_0\geqq|\alpha_{k_1}|\geqq\cdots\geqq|\alpha_{k_{8M-1}}|$ be the $|\alpha_k|$ arranged in decreasing order. Then $j|\alpha_{k_j}|^2\leqq m/8M$ for all j. Hence if $\cos 2\pi k_j n/8M>0$, $j=1, \cdots, q$, we have

$$h(n)\geqq\alpha_0^4+\sum_{j=1}^{q}|\alpha_{k_j}|^4\cos 2\pi k_j n/8M$$
$$-(m/8M)^2\sum_{j=q+1}^{8M-1}1/j^2>(m/8M)^4-(m/8M)^2/q\geqq0.$$

Thus the numbers $\lambda_1=k_1/8M, \cdots, \lambda_q=k_q/8M$ satisfy the conditions of the lemma.

Lemma II. If $n_1, n_2, \cdots$ is a sequence of different integers such that $|n_m|<Am$ for some integer A, and $q\geqq(8A)^2$, then there exist q numbers $\lambda_1, \cdots, \lambda_q$ in $0\leqq\lambda\leqq1$ such that any integer n for which $\cos 2\pi\lambda_j n>0$, $j=1, \cdots, q$, is of the form $n=n_p+n_r-n_s-n_t$. Proof. For every $M=Am>q$, lemma I yields certain numbers $\lambda_1^{(M)}, \cdots, \lambda_q^{(M)}$. For a suitable sequence of M's these numbers will converge towards numbers $\lambda_1, \cdots, \lambda_q$ satisfying the conditions of the lemma.

Lemma III. Let $\tau_1, \tau_2, \cdots$ be a sequence of real numbers such that $|\tau_m|<Hm$ and $|\tau_n-\tau_m|>\alpha>0$ for $n\neq m$ and let $\epsilon>0$ be given. Then there exist numbers $\mu_0, \cdots, \mu_q$ such that any τ satisfying the inequalities $|\mu_0\tau|<\pi/4, \cdots, |\mu_q\tau|<\pi/4$, mod 2π, differs by less than ϵ from some $\tau_p+\tau_r-\tau_s-\tau_t$. Proof. This is an easy consequence of lemma II.

The statement on the translation numbers of an almost periodic function made at the beginning is an easy consequence of lemma III. It will be seen that Bogolioùboff's proof contains ingredients well-known from other proofs of the main theorems on almost periodic functions, in particular, the two convolutions, but they are applied only for periodic functions of an integral variable (for which the theory of "Fourier series" is trivial). The proof shows that in the definition of almost periodicity the relative density of translation numbers may be replaced by the weaker condition that to any $\epsilon>0$ there exists a sequence of translation numbers belonging to ϵ which satisfies the conditions of lemma III. The equivalence of this weaker condition with the relative density was proved without recourse to Bogolioùboff's proof by Følner [Mat. Tidsskr. B **1944**, 24–27 (1944); these Rev. **7**, 60]. *B. Jessen* (Copenhagen).

Blanuša, Danilo. Die Umkehrung der Orthogonalisierungsformel. Bull. Intern. Acad. Croate. Cl. Sci. Math. Nat. **35**, 100–102 (1945).

There exists a familiar procedure for orthogonalizing a system of complex-valued linearly independent functions $f_1, f_2, \cdots$ with inner product $f_{ik}=(f_i, \bar f_k)$. The author solves the inverse problem of converting an orthonormal set into a nonorthogonal set $\{f_k\}$ with preassigned f_{ik}. A geometric intepretation is given. *H. Pollard* (Ithaca, N. Y.).

Blanuša, Danilo. The inversion of the orthogonalization formula. Rad Hrvatske Akademije Znanosti i Umjetnosti. Razred Mat.-Prirodoslov. **86**, 62–74 (1945). (Croatian)

A summary is reviewed above.

Minakshisundaram, S. A uniqueness theorem for eigenfunction expansions. Proc. Nat. Acad. Sci. U. S. A. **33**, 76–77 (1947).

Let $\{w_n(x, y)\}$ be the complete orthonormal set of eigenfunctions of the boundary value problem $\nabla^2w+\mu w=0$, $w(x, y)=0$ on $\dot C$, where C is the boundary of a regular region R, the eigenvalues μ_n being arranged in nondecreasing order. The following theorem is stated. Let $\{a_n\}$ be a sequence of numbers. Suppose there exists a function φ continuous on $R+C$, and vanishing on C, so that

$$\int\int_R\varphi(x, y)w_n(x, y)dxdy=a_n/\mu_n.$$

If $\lim_{t\to0}4\sum a_nw_nJ_1^2(\mu_n^{\frac12}t)/(\mu_nt^2)=f(x, y)$, the series on the left converging for $t>0$, then $f(x, y)\sim\sum a_nw_n$. The general outline of the proof is indicated; it depends on the use of subharmonic functions. *H. Pollard* (Ithaca, N. Y.).

Bourgin, D. G. A class of sequences of functions. Trans. Amer. Math. Soc. **60**, 478–518 (1946).

A sequence $\{a_n\}$ in l^2 is said to be in K if $\sum_{k=0}^{\infty}a_ka_{mk/n}=\delta_{mn}$; in K' if also $\{a_n\}\epsilon l^1$; in K^U if it is in K and $\sum a_nn^{iy}$ converges uniformly. The function $f(x)\sim\sum a_n\sin nx$ is said to belong to K (K', K^U) according as $\{a_n\}$ is in K (K', K^U), and similarly for $\varphi(s)\sim\sum a_nn^{-s}$. The purpose of the paper is to clarify the relationships among these classes of sequences and of functions. In particular, many results obtained earlier for K' are extended to K^U [Bourgin and Mendel, same Trans. **57**, 332–363 (1945); these Rev. **6**, 266]. The results of the present paper were abstracted earlier [Proc. Nat. Acad. Sci. U. S. A. **32**, 1–5 (1946); these Rev. **7**, 294].

The early sections of the paper discuss conditions for membership in K^U and K. For example, $\varphi(s)\epsilon K^U$ if and only if (a) $\varphi(s)$ is meromorphic, (b) $\varphi(s)$ admits a Dirichlet expansion converging uniformly in $\Re s\geqq0$, (c) $\varphi(s)\varphi(-s)=1$. If $\varphi(s)\epsilon K$, then $\varphi(s)$ is regular for $\Re s>0$ with a Dirichlet series converging uniformly there, and $|\varphi(s)|\leqq1$. By use

of these results the following is proved. If $\varphi(z)\epsilon K$ and (a) $\varphi(z)$ is uniformly continuous in $\Re z\geqq 0$, (b) $|\varphi(it)|=1$, (c) $\varphi(z)$ has no zeros in $\Re z>0$ and (d) $\varphi(z)$ has a non-vanishing term in its Dirichlet expansion, then $\varphi(z)=\pm 1$.

A subset S of the integers is said to have a base $\{p_j\}$ of primes if every element of S is of the form $\prod_{j=1}^{N} p_j^{l_j}$. If a_n vanishes except for n in a set with the base $\{p_j\}$, we say that $\{a_n\}$ has base $\{p_j\}$. If $\varphi(z)=\sum a_n n^{-z}$ belongs to K^U, and a_n has a finite base, then $\varphi(z)$ is of the form

$$M^{-z}\left\{1+\sum_{j=1}^{N} h_j n_j^{z}\right\}\Big/\left\{1+\sum_{j=1}^{N} h_j n_j^{-z}\right\},$$

where $n_1, n_2, \cdots$ is an ascending sequence of integers, M is a multiple of the least common multiple of the n_j, $\{h_j\}$ is a real finite sequence and the denominator does not vanish for $\Re z\geqq 0$.

The latter part of the paper supplies proofs of theorems concerning the completeness of sets of functions $\{f(nx)\}$. The theorems were announced earlier in the abstract cited above. *H. Pollard* (Ithaca, N. Y.).

Bary, N. Sur les bases dans l'espace de Hilbert. C. R. (Doklady) Acad. Sci. URSS (N.S.) **54**, 379–382 (1946).

Let $\{\psi_n\}$ and $\{g_n\}$ form a biorthogonal set of functions on (a, b); they are assumed to be in $L^2(a, b)$. The set $\{\psi_n\}$ is called a Bessel system if $\sum c_n^2<\infty$ converges for every $f(x)$ in L^2, where (*) $c_n=(f, g_n)$. The set $\{\psi_n\}$ is a Hilbert system if for every sequence $\{c_n\}$ in l^2 there is an $f(x)$ in L^2 for which (*) holds. A set $\{\psi_n\}$ which has both properties is called a Riesz-Fischer system. Theorems of the following kind are stated without proof. (I) The set $\{\psi_n\}$ is a Hilbert system if and only if the infinite matrix $\|(\psi_n, \psi_k)\|_{n,k}$ is bounded. (II) The set $\{\psi_n\}$ is a Bessel system if and only if $\|(g_n, g_k)\|_{n,k}$ is bounded. (III) If $\{g_n\}$, $\{\psi_n\}$ are complete and one is a Riesz-Fischer system, then both systems are bases. (IV) If $\{\psi_n\}$ is a base and a Riesz-Fischer system, then there exist $m>0$, $M>0$ such that $m\|f\|\leqq(\sum c_n^2)^{\frac{1}{2}}\leqq M\|f\|$ for every f in L^2. *H. Pollard* (Ithaca, N. Y.).

Gross, Bernhard, and Levi, Beppo. On the calculation of the inverse Laplace transformation. Math. Notae **6**, 213–224 (1946). (Spanish)

Let $f(s)$ have both representations

$$f(s)=\int_0^\infty (s+t)^{-1}d\alpha(t) \tag{1}$$

and

$$f(s)=\int_{0+}^\infty e^{-st}\varphi(t)dt. \tag{2}$$

Then $\varphi(t)$ can be calculated by applying Stieltjes' inversion formula to (1) to obtain $\alpha(t)$; one has $\varphi(t)=\int_0^\infty e^{-tu}d\alpha(u)$. Several special cases are studied. *R. P. Boas, Jr.*

Harmonic Functions, Potential Theory

Walsh, J. L. Note on the critical points of harmonic functions. Proc. Nat. Acad. Sci. U. S. A. **33**, 54–59 (1947).

Generalization of earlier results [same Proc. **33**, 18–20 (1947); these Rev. **8**, 461]. Let R be a multiply connected region and D a part of its boundary. By using the universal covering surface R^∞, inequalities are obtained which define regions containing no critical point either of the harmonic measure $\omega(z, D, R)$ or of the Green's function $g(z, z_0, R)$. *J. Ferrand* (Caen).

Fichera, Gaetano. Sul problema di Dini-Neumann nel piano. Bull. École Polytech. Jassy [Bul. Politehn. Gh. Asachi. Iaşi] **1**, 281–288 (1946).

Let D be a region in the (x, y)-plane bounded by the regular closed curves $C_0, C_1, \cdots, C_p$, where $C_1, \cdots, C_p$ are exterior to each other and contained in C_0. Let $f_i(s)$ be functions of the arc length s on C_i, which are continuous except for a set of linear measure 0. The author considers the problem of finding multiple valued harmonic functions $u(x, y)$ in D for which $\partial u/\partial n=f_i(s)$ on C_i (for $i=0, \cdots, p$) and which change by prescribed "periods" Q_i if one describes a closed path around C_i (for $i=1, \cdots, p$). He finds that the solution u is uniquely determined and that a necessary condition for its existence is the relation

$$\sum_{i=0}^{p}\int_{C_i} f_i(s)ds=0. \tag{1}$$

He shows that, if (1) is satisfied, the determination of u can be reduced completely to the solution of certain Dirichlet problems for the region D. *F. John* (New York, N. Y.).

***Laasonen, Pentti. Über präharmonische Funktionen.** C. R. Dixième Congrès Math. Scandinaves 1946, pp. 108–117. Jul. Gjellerups Forlag, Copenhagen, 1947.

La notion de fonction préharmonique remonte à Wiener [1923] et Bouligand [1925]. Soit, dans le plan, un réseau de carrés; une fonction définie sur un ensemble D de sommets est préharmonique si, en chaque sommet "intérieur" à D, sa valeur est égale à la moyenne de ses valeurs aux 4 sommets voisins. Ces fonctions peuvent servir à l'étude des fonctions harmoniques par un passage à la limite, déjà effectué par Phillips et Wiener [J. Math. Phys. Mass. Inst. Tech. **2**, 105–124 (1923)]. L'auteur se propose d'obtenir, par un tel procédé, la représentation conforme d'un domaine simplement connexe (limité par une courbe de Jordan) sur une bande limitée par deux droites parallèles. Auparavant, il résout le problème de Poisson pour fonction préharmonique (solution immédiate), définit les "dérivées partielles" d'une fonction préharmonique (ces dérivées ne sont pas définies sur le même réseau) et la notion de couple de fonctions préharmoniques associées. Les fonctions préharmoniques minimisent une somme analogue à l'intégrale $\iint\{(\partial u/\partial x)^2+(\partial u/\partial y)^2\}dxdy$ des fonctions harmoniques.

Sur ce sujet des fonctions préharmoniques, l'auteur cite C. Blanc [Comment. Math. Helv. **12**, 153–163 (1939); **13**, 54–67 (1940); ces Rev. **1**, 213; **2**, 293]. Citons aussi J. Ferrand [Bull. Sci. Math. (2) **68**, 152–180 (1944); ces Rev. **7**, 149], qui fait une étude directe, assez poussée, des fonctions préholomorphes èt traite aussi de la représentation conforme par un passage à la limite. *H. Cartan.*

Weiss, P. An extension of Cauchy's integral formula by means of Maxwell's stress tensor. J. London Math. Soc. **21**, 210–218 (1946).

Let u and v be harmonic functions of $(x_1, \cdots, x_m)$ in a domain D and on its boundary S and, with tensor notation, let $T_{jk}(u, v)=u_iv_i\delta_{jk}-(u_jv_k+u_kv_j)$, the subscripts of u and v denoting differentiation. It is shown that

$$\int_S T_{jk}(u, v)n_k dS=0.$$

Because of its connection with Maxwell's stress tensor, the author suggests that the result be called "Maxwell's reciprocal theorem." For $m=2$, the result can be expressed as

$\int_C f(z)g(z)dz=0$, where $f(z)$ and $g(z)$ are analytic functions of the complex variable z in and on the rectifiable curve C. The author gives an extension of Cauchy's integral formula, and applications to spherical harmonics, Green's function, the differential equation $\nabla^2 u+cu=0$ and eigenfunctions.

E. F. Beckenbach (Los Angeles, Calif.).

Pólya, G. Estimating electrostatic capacity. Amer. Math. Monthly **54**, 201–206 (1947).

A short account of results and conjectures due to the author and G. Szegö concerning estimations of the electrostatic capacity of convex bodies and continuing former work of these authors [Amer. J. Math. **67**, 1–32 (1945); Szegö, Bull. Amer. Math. Soc. **51**, 325–350 (1945); these Rev. **6**, 227].

W. Fenchel (Copenhagen).

Reade, Maxwell O. On averages of Newtonian potentials. Bull. Amer. Math. Soc. **53**, 321–331 (1947).

Il est connu [F. Riesz, Acta Math. **54**, 321–360 (1930)] que chaque fonction surharmonique U définit une distribution de masses positives μ, telle que, si on restreint μ à un ensemble ouvert borné quelconque, le potentiel de la distribution obtenue soit, dans l'ouvert, égal à U à l'addition près d'une fonction harmonique. En outre, toute moyenne effectuée sur U et ses translatées $V(x)=\int U(x-y)d\lambda(y)$ (λ distribution positive de masse totale 1) donne naissance à une fonction V surharmonique, et la distribution ν associée à V est donnée par la même moyenne $\nu(x)=\int\mu(x-y)d\lambda(y)$. C'est ce dernier point que l'auteur se propose de prouver; dans ce but, il fait les hypothèses restrictives (inutiles) que voici: 1° il se place dans le plan; 2° au lieu de fonctions surharmoniques, il se borne à envisager des potentiels de masses positives; 3° il suppose que μ est portée par un ensemble borné; 4° il semble supposer que μ est définie par une densité par rapport à l'aire; 5° il suppose que λ est définie par une densité ne prenant que les valeurs 0 et 1.

H. Cartan (Strasbourg).

Deny, Jacques, et Lelong, Pierre. Sur une généralisation de l'indicatrice de Phragmen-Lindelöf. C. R. Acad. Sci. Paris **224**, 1046–1048 (1947).

Au lieu de considérer une fonction $\log|f(z)|$, où f est holomorphe dans une bande ou un angle, les auteurs envisagent une fonction sousharmonique (sh) dans un cône ou un cylindre de l'espace R^p à p dimensions; le cas $p=2$ a été étudié par G. H. Hardy et W. Rogosinski [Proc. Roy. Soc. London. Ser. A. **185**, 1–14 (1946); ces Rev. **7**, 448]. Voici le théorème dans le cas d'un cylindre Γ, produit d'un domaine D de R^{p-1} par l'intervalle $0<x<\infty$: soit u sh dans Γ, et $u(m, x)\leqq Ae^{+\rho x}$ ($\rho\geqq 0$, $m\epsilon D$); soit $\varphi(m)=\lim\sup_{x\to\infty} u(m, x)e^{-\rho x}$, $\varphi^*(m)=\lim\sup_{m'\to m}\varphi(m')$ (régularisée supérieure de φ); soit D' un domaine compact, régulier, contenu dans D, et dont la constante fondamentale (plus petite valeur propre de l'équation $\Delta f+\lambda^2 f=0$ pour D') soit supérieure à ρ; alors φ^* est majorée, dans D', par la solution f de $\Delta f+\rho^2 f=0$ relative à la donnée frontière φ^*; interprétation: la fonction $f(m)e^{\rho x}$, harmonique dans le cylindre ($m\epsilon D'$, $0<x<\infty$), est la meilleure majorante harmonique "à l'infini" de $u(m, x)$. Théorème analogue lorsque Γ est un cône au lieu d'un cylindre.

Les auteurs indiquent quelques étapes de la démonstration, et signalent à cette occasion une généralisation de la notion de fonction sousmédiane: $v(m)$ sommable dans D ouvert est de classe $\mathcal{C}_\alpha$ si, pour chaque point m_0 de D', on a, pour tout $r>0$ assez petit, $v(m_0)\leqq\alpha(r)+$moyenne de v sur la boule de centre m_0 et de rayon r; $\alpha(r)$ désigne une fonction croissante donnée, définie pour $r>0$, telle que $\lim_{r\to 0}\alpha(r)=0$. Si φ est de classe $\mathcal{C}_\alpha$, φ est égale à sa régularisée supérieure φ^* sauf sur un ensemble de mesure nulle; toute famille de fonctions de classe $\mathcal{C}_\alpha$, bornée supérieurement, a pour borne supérieure une fonction de classe $\mathcal{C}_\alpha$.

H. Cartan.

Massonnet, Charles. Sur le problème aux limites fondamental relatif aux fonctions biharmoniques. Bull. Soc. Roy. Sci. Liége **14**, 308–312 (1945).

Differential Equations

✱Kamke, E. Differentialgleichungen reeller Funktionen. Chelsea Publishing Company, New York, N. Y., 1947. xii+436 pp. $3.95.

Photographic reproduction of the edition of 1930, published by Akademische Verlagsgesellschaft, Leipzig.

✱Collatz, Lothar. Eigenwertprobleme und ihre numerische Behandlung. Mathematik und ihre Anwendungen in Physik und Technik, Reihe A, Band 19. Akademische Verlagsgesellschaft, Leipzig, 1945. xiii+338 pp.

In summarizing this book we may conveniently consider it as divided into three parts. Part I, covering pages 5 to 53, presents an interesting and valuable collection of practical applications, drawn mostly from the field of engineering, which lead to boundary value problems for ordinary and partial differential equations. A tabular list of about one hundred such problems with the associated differential equation and boundary conditions is furnished for reference and comparison. Typical of the problems in the list are transverse oscillations of strings, beams, membranes and plates, deflection of beams and plates, buckling of loaded beams and plates, oscillations of rotating shafts, etc.

Part II, covering pages 53 to 177, deals with the mathematical theory of boundary value problems. Here the existence, reality and distribution of characteristic numbers are investigated; the orthogonality or biorthogonality of the characteristic functions is developed; many illustrative examples are worked. The Green's function is then defined and its properties and use explained. A reference list of Green's functions for various different differential systems makes the subject unusually lucid.

The relation between a differential equation with boundary conditions and the corresponding integral equation is next considered, together with some properties of integral equations and an investigation of the asymptotic distribution of the characteristic numbers. The minimizing properties of the characteristic numbers are developed, comparison theorems for the approximate location of characteristic numbers are established and the Fourier expansion of arbitrary functions in series of characteristic functions is treated in considerable detail. An extensive list of boundary value problems that can be solved explicitly adds materially to the value of the book.

Part III, pages 177 to 329, takes up various methods of numerical solution of boundary value problems. These include step by step approximations, graphical integration, the Rayleigh-Ritz method and methods depending on finite

differences. Here, as throughout the book, the theory is kept in close touch with practice by numerous specific examples.
W. E. Milne (Corvallis, Ore.).

★**Pleijel, Åke. Quelques remarques sur certains problèmes de vibrations dégénérés.** C. R. Dixième Congrès Math. Scandinaves 1946, pp. 54–58. Jul. Gjellerups Forlag, Copenhagen, 1947.

The paper deals with the differential system

$$Fu=\lambda Gu, \quad A_iu=0, \qquad i=0, 1, \cdots, 2m-1,$$

in which λ is a parameter,

$$Fu=\sum_{r=0}^{m}[f_r(x)u^{(r)}]^{(r)}, \quad Gu=\sum_{r=0}^{n}[g_r(x)u^{(r)}]^{(r)},$$

with $m>n$, and the A_iu are linear forms in the values of $u, u', \cdots, u^{(2m-1)}$ at $x=a$ and $x=b$. With $F(u, v)$ defined by the relation $F(u, v)=\int_a^b Fu\cdot v dx$, such systems have been studied by Kamke under the hypotheses that $(Fu, v)=(u, Fv)$, $(Gu, v)=(u, Gv)$ and $(Fu, v)\geqq 0$, and that they are normal in the sense that either $\lambda=0$ is not a characteristic value or every characteristic solution u_0 associated with $\lambda=0$ fulfills the relation $(Gu_0, u_0)\neq 0$.

Under the assumption that the two equations $Fu=0$, $Gu=0$, together with the boundary conditions, imply $u\equiv 0$, the paper deduces a condition by the adjunction of which $\lambda=0$ may be removed from the set of characteristic values and a nonnormal system may be accordingly normalized.
R. E. Langer (Madison, Wis.).

Kitamura, Taiiti. Some inequalities on a system of solutions of linear simultaneous differential equations. Tôhoku Math. J. 49, 308–311 (1943).

Consider the linear system $dx/dt=A(t)x$, where x is an n-dimensional column vector and $A(t)$ an $n\times n$ matrix. Define

$$\|x\|=\left\{\sum_{i=1}^{n}|x_i|^p\right\}^{1/p}, \quad p\geqq 1, \qquad \|A\|=\max_{\|x\|>0}\|A(t)x\|/\|x\|,$$

where $x_1, \cdots, x_n$ are the components of x. The author proves that, for $t\geqq t_0$,

$$\|x(t_0)\|\exp\left\{-\int_{t_0}^{t}\|A\|dt\right\}\leqq\|x(t)\|\leqq\|x(t_0)\|\exp\left\{\int_{t_0}^{t}\|A\|dt\right\},$$

with equality occurring only if $A(t)$ is a diagonal matrix. A similar but more complicated inequality is obtained for the equation $dx/dt=A(t)x+b(t)$. *R. Bellman.*

Giuliano, L. Generalizzazione di un lemma di Gronwall e di una diseguaglianza di Peano. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 1, 1264–1271 (1946).

The author proves, under certain conditions too detailed to include here, that if

$$||u(x)|-|u(x_0)||\leqq\int_{x_0}^{x}\phi(x_1, |u(x_1)|)dx_1,$$

and v_1, v_2 represent respectively the solutions of

$$y(x)=\pm\int_{x_0}^{x}\phi(x_1, y(x_1))dx_1+|u(x_0)|,$$

then $v_2\leqq|u(x)|\leqq v_1$, $x\geqq x_0$. This is a generalization of various results due to Gronwall [Ann. of Math. (2) 20, 292–296 (1919)], Peano [Nouv. Ann. Math. (3) 11, 79–82 (1892)] and others. *R. Bellman* (Princeton, N. J.).

Magenes, E. Sopra un problema di T. Satô per l'equazione differenziale $y''=f(x, y, y')$. I. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 2, 130–136 (1947).

The author considers the problem of the existence of solutions of the equation $y''=f(x, y, y')$, passing through a given point and tangent to a given curve. *R. Bellman.*

Miranda, C. Problemi ai limiti per le equazioni differenziali ordinarie del secondo ordine in forma parametrica. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 2, 164–169 (1947).

Let C be a curve parametrized according to arc length and let θ be the angle between the tangent to the curve at a point and the x-axis. Let α, β be the direction cosines of the tangent. Consider the differential equation $d\theta/ds=f(x, y, \alpha, \beta)$, or in Cartesian coordinates

$$\frac{x'y''-x''y'}{(x'^2+y'^2)^{3/2}}=f(x, y, x'(x'^2+y'^2)^{-\frac{1}{2}}, y'(x'^2+y'^2)^{-\frac{1}{2}}).$$

The author considers the problem of determining solutions of the above equations passing through two given points.
R. Bellman (Princeton, N. J.).

Tricomi, Francesco. Sulla funzione di Green di un'equazione differenziale decomposta in fattori simbolici. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. 80, 159–183 (1945).

The author proves that, if $G(x, y)$ and $H(x, y)$ are, respectively, the Green's functions of the ordinary linear homogeneous differential equations $L(u)=0$, $M(u)=0$, of orders $n, m\geqq 2$, subject to Sturm-Liouville conditions at the endpoints of the interval $[a, b]$, then the composed function $J(x, y)=\int_a^b G(x, z)H(z, y)dz$ is the Green's function of the equation $M(L(u))=0$, with appropriate boundary conditions. The results are extended to partial differential equations of the fourth order and application is made to the study of the vibration of elastic plates. *R. Bellman.*

Mangeron, Dumitru Ion. On a class of differential equations of higher order. Gaz. Mat., București 52, 207–212 (1947). (Romanian. French summary)

L'auteur énonce quelques résultats préliminaires concernant des conditions pour qu'une équation différentielle linéaire homogène, d'ordre quelconque, puisse être transformée, par un choix convenable d'une transformation de la variable indépendante, dans une équation à coefficients constants choisis à volonté. *From the author's summary.*

Blanc, Charles. Sur les équations différentielles linéaires non homogènes, à coefficients variables. Ann. Univ. Grenoble. Sect. Sci. Math. Phys. (N.S.) 22, 119–134 (1946).

The ordinary differential equation

$$Du=\sum_{0}^{N}a_i(t)u^{(i)}(t)=F(t)$$

is studied with a view to obtaining a solution in terms of $F(t)$ and its successive derivatives. This form of solution is somewhat in contrast with that obtained by the method of variation of parameters, where $F(t)$ occurs in the integrand of a definite integral. The coefficients in Du are real functions of the real variable t and are either continuous or have at most discontinuities of the first kind; $F(t)$ may be complex but is continuous with derivatives of all orders. The solution is first found as a definite integral with remainder

term after the manner of Taylor's formula. Passage to the limit yields a series of the desired type. Comparison is made between the solution of the given equation and that of an equation whose coefficients are constants equal to the given coefficients at a point in whose neighborhood the study is made. A limit is obtained for the difference between these two solutions. Some consideration is given to the asymptotic behavior of the solutions. *W. M. Whyburn.*

Carafa, M. Risoluzione effettiva, mediante integrali definiti, dell'equazione differenziale binomia:

$$\frac{d^n y}{dx^n}-a(x)y=f(x).$$

Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 2, 152–158 (1947).

This is a brief statement, without proofs, of some results which are to be published more completely elsewhere. The author considers the equation of the title and the initial conditions $y(x_0)=y_0$, $y'(x_0)=y_0^{(1)}$, $\cdots$, $y^{(n-1)}(x_0)=y_0^{(n-1)}$, assuming that the functions $a(x)$ and $f(x)$ are analytic and regular in the neighborhood of x_0. He obtains, in the usual way, a Volterra integral equation of the second kind which is satisfied by $y(x)$; he proceeds to solve this equation by a complicated operational method which is related to Fantappiè's theory of analytic functionals. His final conclusion is that $y(x)$ can be represented explicitly by an expression which involves the given functions $a(x)$ and $f(x)$, a certain universal function $M_n(x, \xi, z, \mu, \omega, c, \theta)$, and seven operations of integration. *L. A. MacColl* (New York, N. Y.).

Rosenblatt, Alfred. On the developments in series of the solutions of differential equations of the first order and first degree in the neighborhood of an essential singular point. Bol. Fac. Ingen. Montevideo 3 (Año 10), 127–143 (1946). (Spanish)

The paper also appeared in Facultad de Ingeniería Montevideo. Publ. Inst. Mat. Estadística 1, 111–127 (1946); these Rev. 8, 71.

Germay, R.-H.-J. Sur l'emploi d'une intégrale $(p-k)^{\text{ième}}$ pour la réduction d'un système complètement intégrable d'équations aux différentielles totales. Bull. Soc. Roy. Sci. Liége 15, 167–176 (1946).

Let

$$(*)\qquad dz_i=\sum_{j=1}^{n}a_{ij}(x_1, \cdots, x_n; z_1, \cdots, z_p)dx_j, \quad i=1, \cdots, p,$$

be a completely integrable pth order system of total differential equations. The paper shows that the knowledge of $p-k$ integrals of (*) can be used to reduce this system to a completely integrable kth order system. This is a generalization of an earlier paper by the author [Ann. Soc. Sci. Bruxelles. Sér. I. 60, 86–92 (1946); these Rev. 8, 74]. *F. G. Dressel* (Durham, N. C.).

Comét, Stig. Sur certains systèmes d'équations aux dérivées partielles. C. R. Acad. Sci. Paris 224, 1045–1046 (1947).

The following theorem is announced. Let Ω be the $p\times p$ operator matrix $(\partial/\partial x_{jk})$, where the x_{jk} are independent variables, Δ its determinant, Δ_{jk} the cofactor of $\partial/\partial x_{jk}$ and Ω_i the adjoint of Ω. These operators are applied to power series in the variables. Let $y=(y_1, \cdots, y_p)$ be a vector whose components are such functions. Suppose that $\Delta y=0$. Then there exists a function ϕ such that $\Delta\phi=0$ and $y_j=\Delta_{j1}\phi$ $(j=1, \cdots, p)$. An expression for the most general solution of $\Delta\phi=0$ is given and an application to functions of quaternions is indicated. *O. Todd-Taussky* (London).

Wintner, Aurel. On the momentum operator in wave mechanics. Physical Rev. (2) 71, 547–549 (1947).

With a view to applications in wave mechanics, the author proves the following theorem. Let $f(x, y, z)$ be a bounded continuous function and let $\varphi(x, y, z)$ possess continuous partial derivatives up to the second order and satisfy the differential equation $\nabla^2\varphi+f(x, y, z)\varphi=0$. If φ is of integrable square over the whole space, then so its vector gradient $\nabla\varphi$. He then shows that $f(x, y, z)$ need only be assumed to be bounded above, and that both results can be extended to the equation $\nabla^2\varphi+f(x, y, z)\varphi=g(x, y, z)$, where $g(x, y, z)$ is an arbitrary continuous function of integrable square. An example is given to show that the boundedness condition on f cannot be dispensed with altogether. The proofs are of an elementary Tauberian character. *F. Smithies* (Cambridge, England).

Ambarzumian, G. A. Stochastic processes with two parameters giving in infinity the normal correlation. Acad. Sci. Armenian SSR. Proc. [Doklady] 5, 65–70 (1946). (Russian. Armenian and English summaries)

In analytical terms the problem of the paper is the determination of a solution $p(t, x, y)$ of the parabolic differential equation

$$p_t=b_1^2p_{xx}+2rb_1b_2p_{xy}+b_2^2p_{yy}-[(a_1x+c)p]_x-[(a_2y+c_2)p]_y$$

with the initial condition $p(0, x, y)=p_0(x, y)$, where $p_0(x, y)$ is a given density function. Here a_i, b_i, c_i and r are real constants, $|r|<1$. An explicit solution is obtained in terms of Hermite polynomials. As $t\to\infty$, $p(t, x, y)$ approaches a normal density function with correlation coefficient r. *W. Feller* (Ithaca, N. Y.).

Rubinstein, L. I. On the solution of Stefan's problem. Bull. Acad. Sci. URSS. Sér. Géograph. Géophys. [Izvestia Akad. Nauk SSSR] 11, 37–54 (1947). (Russian. English summary)

A method for the solution of Stefan's problem for the case of a finite linear conductor is given. The equations of the problem are $u_{xx}=u_t$, $0<x<y(t)$; $v_{xx}=a^{-2}v_t$, $y(t)<x<1$; $u(0, t)=f_1(t)$, $u(y(t), t)=v(y(t), t)=0$, $v(1, t)=f_2(t)$; $u(x, 0)=\varphi_1(x)$, $0<x<y(0)$, $v(x, 0)=\varphi_2(x)$, $y(0)<x<1$. The method consists in first applying a nonlinear transformation to the coordinates, which transforms the regions of definition of each of the phase functions into the interval $(0, 1)$. A system of integro-differential equations is set up and solved by successive iterations. A proof for the convergence of the iteration process is given and the uniqueness of the solution is established. The author mentions that his procedure seems to be applicable only to the linear case or to those reducible to it, such as the spherical or cylindrical cases with radial or axial symmetry. *H. P. Thielman* (Ames, Iowa).

Finzi-Contini, Bruno. La similitudine nei campi armonici con dispersione sul contorno. Ist. Lombardo Sci. Lett. Cl. Sci. Mat. Nat. Rend. (3) 4(73), 699–715 (1940).

This is an investigation of the conditions under which the lines of heat flux in two geometrically similar heat conducting solids will be geometrically similar. The condition is that the dimensionless quantity $F=\mu/(L\nu)$ (L, characteristic

length; μ, heat conduction coefficient of the solid; ν, coefficient of heat transmission between the body and the surrounding medium) is the same for model and prototype.
P. Neményi (Washington, D. C.).

Difference Equations, Special Functional Equations

Ionesco, D. V. Quelques problèmes relatifs à une formule de récurrence. Disquisit. Math. Phys. 4, 3–94 (1945).

The author solves the equation

$$u_{m,n}=au_{m-1,n}+bu_{m,n-1}+cu_{m-1,n-1}$$

(a, b, and c are constants) for two different kinds of boundary conditions: (i) when the values of $u_{m,0}$ and $u_{0,n}$ are prescribed, (ii) when the values of $u_{m,0}$ and $u_{n,n}$ are prescribed. The solutions are applied to the determination of the coefficients in the power series expansion of $1/(1-ax-by-cxy)$ and to the determination of a function $f(y)$ such that the power series development of $f(y)/(1-ax-by-cxy)$ shall have zero coefficients for xy, x^2y^2, x^3y^3, $\cdots$; in the latter problem it is shown that the only function $f(y)$ possessing this property is $(a-2aby-bcy^2)/(a+cy)$. The second part of the paper obtains solutions for the equation

$$u_{m,n,p}=au_{m-1,n,p}+bu_{m,n-1,p}+cu_{m-1,n-1,p}+a'u_{m-1,n,p-1}+b'u_{m,n-1,p-1}+c'u_{m-1,n-1,p-1}+ku_{m,n,p-1}$$

with prescribed boundary values $u_{m,n,0}$, $u_{m,0,p}$, and $u_{0,n,p}$. This solution is applied to the determination of the coefficients in the power series expansion of

$$1/(1-ax-by-kz-a'xz-b'yz-cxy-c'xyz).$$

D. Moskovitz (Pittsburgh, Pa.).

Stone, William M. A note on a paper by Faust and Beck. J. Appl. Phys. 18, 414–416 (1947).

P. Samuelson's generalized Laplace transformation [Bull. Amer. Math. Soc. 52, 240 (1946)] is applied to find the solution of a pair of simultaneous difference equations with constant coefficients which arise in electric network theory.
A. E. Heins (Pittsburgh, Pa.).

Andronov, A. A., and Maier, A. G. The simplest linear systems with retardation. Avtomatika i Telemehanika 7, 95–106 (1946). (Russian)

The authors consider the retarded differential equation

$$(1)\qquad dx/dt=a_1x+a_2x(t-\tau),$$

where α, β and τ are real and $x(t)$ is known in the interval $0\leqq t\leqq\tau$. It follows from a theorem of Hilb [Math. Ann. 78, 137–170 (1917)] that every solution of (1) may be expanded in the form (2) $\sum_n c_ne^{\lambda_nt}$, c_n constant, where the λ_n are the zeros of the equation (3) $\lambda=a_1+a_2e^{-\lambda\tau}$, provided these zeros are simple. Thus the behavior of the solutions of (1) as $t\rightarrow+\infty$ is determined by the location of the zeros of (3). For physical applications it is important to know when all solutions tend to zero. This requires that $\Re(\lambda_n)<0$. The authors investigate the regions of the (a_1, a_2) plane for which this is true. The equation

$$(4)\qquad dx/dt=a_1x+a_2x(t-\tau)+A\cos(\gamma t+\phi)$$

is also considered. Some helpful graphs and tables are given.
R. Bellman (Princeton, N. J.).

Truesdell, C. On the functional equation

$$\frac{\partial}{\partial z}F(z,\alpha)=F(z,\alpha+1).$$

Proc. Nat. Acad. Sci. U. S. A. 33, 82–93 (1947).

The author formulates in fifteen theorems results on the solutions of the differential-difference equation of the title. These results include power series, Newton series, contour integrals for the solutions and infinite series and integrals involving the solutions. He lists particular solutions of the functional equation involving Bessel functions, Legendre functions, Laguerre and Hermite functions, confluent hypergeometric functions, and mentions others containing some functions occurring in number theory or mathematical statistics. The application of the general results to these particular solutions lead to a great many formulae (some known, some new) for each of the functions concerned; the author gives about forty examples. *A. Erdélyi.*

Myller, A. Equations itérales linéaires du second ordre à coefficients constants. Bull. École Polytech. Jassy [Bul. Politehn. Gh. Asachi. Iaşi] 1, 270–273 (1946). (Romanian. French summary)

On peut obtenir des solutions $y=f(x)$ d'une équation itérale linéaire du second ordre à coefficients constants en partant de l'observation qu'elles restent inalterées quand on échange x et y respectivement en y et $py+qx$. On calcule effectivement des solutions algébriques d'ordre 1 et 2.
From the author's summary.

Kitamura, Taiiti. On the solution of some functional equations. Tôhoku Math. J. 49, 305–307 (1943).

Let $g(x)$ be a transformation defined on the x-axis. Let there exist sets D_k $(-\infty<k<+\infty)$ such that D_{k+1} is the one-to-one image of D_k under the transformation. Then there exists a solution $\varphi(x)$ of the functional equation $F(\varphi(g(x)), \varphi(x), x)=0$ in $\sum_{k=-\infty}^{\infty}D_k$, which takes arbitrarily prescribed values in D_0. Here the first derivatives of F with respect to its first two arguments are assumed not to vanish.
F. John (New York, N. Y.).

Integral Equations

Khalilov, Z. I. Sur l'équation intégrale de Fredholm à noyau linéaire par rapport au paramètre. C. R. (Doklady) Acad. Sci. URSS (N.S.) 54, 567–569 (1946).

The author studies the equation

$$(1)\qquad u(x)=\int_a^b\{K_0(x,s)+\lambda K_1(x,s)\}u(s)ds,$$

where $f(x)$ is L and K_0, K_1 are L_2 in each variable separately. In accordance with J. D. Tamarkin [Ann. of Math. (2) 28, 127–152 (1927)] it is known that either (1) has no solutions for all f, whatever λ may be (the singular case), or (1) has a unique solution for all f and λ except, perhaps, for a denumerable infinity of values λ with no finite limiting points (nonsingular case). The author states (mostly without detailed proofs) a number of results, of which we cite the following. In the nonsingular case (1) is equivalent to (2) $v(x)=\lambda\int K(x,s)v(s)ds+f(x)$, where $K(x,s)=K_1(x,s)+\int K_1(x,t)R(t,s)dt$; $R(x,s)$ is the resolvent of K_0. If K_0, K_1 are symmetric and K_1 is definite, then $K_0+\lambda K_1$ has at least one characteristic value; its

spectrum is the λ-plane or consists of simple real characteristic values. If K is symmetric, then for λ not a characteristic value the solution of (1) is expressible in the form $u(x)=u_0(x)-\lambda\sum(\lambda-\lambda_\nu)^{-1}f_\nu u_\nu(x)$, where $u_0(x)$ is the solution of $u(x)=\int K_0(x, s)u(s)ds+f(x)$, $f_\nu=a_\nu-b_\nu$, the a_ν, b_ν being the Fourier constants of $f(x)$, $\int K_0(x, s)f(s)ds$ with respect to the characteristic functions of $K_0+\lambda K_1$. Necessary and sufficient conditions are stated for the existence of a solution of (1) when $\lambda=\lambda_{p+1}=\cdots=\lambda_{p+q}$ are the characteristic values of the nonlinear singular kernel $K_0+\lambda K_1$.

W. J. Trjitzinsky (Urbana, Ill.).

Parodi, Maurice. Sur deux applications de la transformation de Laplace. C. R. Acad. Sci. Paris **224**, 996–998 (1947).

The author shows that the general form of the kernel $K(x)$ for which the equation $\int_0^\infty K(x)f(x+t)dx=g(t)$ has a solution of the form

$$f(t)=\int_0^\infty K(x)g^{(n)}(x+t)dx, \qquad n>1,$$

is $K(x)=\pm(-1)^{n/2}x^{n/2-1}/\Gamma(n/2)$. It is also shown that, if $K(x,t)=t^{\nu/2}x^{-\nu/2}J_\nu(2(xt)^{\frac{1}{2}})$, the equation $\int_0^\infty K(x,t)f(x)dx=g(t)$ has a solution of the form $f(t)=\int_0^\infty K(x, t)g(x)dx$.

H. R. Pitt (Belfast).

Parodi, Maurice. Application d'une séquence symbolique à la résolution d'équations intégrales. Bull. Sci. Math. (2) **70**, 122–127 (1946).

The Laplace integral is applied to the formal solution of several integral equations. The particular symbolic sequence used here is the following: if $f(t)\supset\varphi(p)$ and $g(t)\supset p^{\frac{1}{2}}f(1/p)$ then $\frac{1}{2}\pi^{\frac{1}{2}}\tan(t^2/4)\supset\varphi(p^2)$.

H. Pollard (Ithaca, N. Y.).

✱Hellsten, Ulf. Determination of the denominator of Fredholm in some types of integral equations. C. R. Dixième Congrès Math. Scandinaves 1946, pp. 118–122. Jul. Gjellerups Forlag, Copenhagen, 1947.

The author determines explicitly the eigenvalues of the integral equation

$$\varphi(x)=\lambda\int_0^1 K(x, y)\varphi(y)dy,$$

where $K(x, y)$ $(0\leqq x\leqq 1,\ 0\leqq y\leqq 1)$ is 0 above the line $y=bx+a$ $(0<a\leqq 1,\ 0\leqq(1-a)/b<1)$ and 1 below it. It is indicated that similar methods could be applied to the case when the straight line $y=bx+a$ is replaced by a curve.

M. Kac (Ithaca, N. Y.).

Kostitzin, Vladimir. Sur une généralisation de l'équation intégrale d'Abel. C. R. Acad. Sci. Paris **224**, 885–887 (1947).

The author shows that a general solution of the equation

$$\phi(x)=f(x)+\sum_{h=0}^{n-1}\frac{\lambda_h}{\Pi(-h/n)}\int_0^x\phi(z)(x-z)^{-h/n}dz,$$

in which $\Pi(s)=\Gamma(s+1)$, can be found in the form

$$\phi(x)=f(x)+\sum_{i=0}^{n-1}L_iF_{-i/n}(x)+\sum_{k=1}^{n}\sum_{i=0}^{n-1}M_{ik}\int_0^x e^{\rho_k z}F_{-i/n}(x-z)dz,$$

where

$$F_s(x)=\{\Pi(s)\}^{-1}\int_0^x(x-u)^sf(u)du.$$

The numbers L_i, M_{ik} can be determined by elementary algebraic methods as the solutions of linear equations with coefficients depending on λ_h and ρ_k, while the numbers ρ_k themselves are the characteristic roots of a system of n linear equations with coefficients depending only on λ_h.

H. R. Pitt (Belfast).

Nazarov, N. N. On one class of homogeneous nonlinear integral equations. Acta [Trudy] Univ. Asiae Mediae. Ser. V-a. Fasc. **28**, 12 pp. (1939). (Russian. English summary)

The author studies the nonlinear homogeneous integral equation of Hammerstein type

$$\varphi(x)=\lambda\int_0^1 K(x, y)f(y, \varphi(y))dy, \tag{1}$$

where the function $f(y, t)$ is given by a convergent series $f(y, t)=t+A_2(y)t^2+\cdots$ $(A_2(y)\neq 0)$; the kernel $K(x, y)$ is regular, definite and symmetric. It is proved that in the vicinity of any characteristic value λ_k, for which $|\lambda_i-\lambda_k|\geqq 1$ $(i\neq k)$, (1) has an analytic solution, not identically zero, of the form

$$\varphi(x)=(\lambda-\lambda_k)f_{k1}(x)+(\lambda-\lambda_k)^2f_{k2}(x)+\cdots.$$

It is indicated that the definite character of $K(x, y)$ is not necessary; it is sufficient to assume that $K(x, y)$ has a finite number only of characteristic values of one sign, the others being of the opposite sign.

W. J. Trjitzinsky.

Nazarov, N. N. Methods for the solution of nonlinear integral equations of Hammerstein's type. Acta [Trudy] Univ. Asiae Mediae. N. S. Fasc. **6**, 14 pp. (1945). (Russian)

This paper is largely in the form of a brief report of methods and results. The author indicates that many boundary value problems for nonlinear differential equations (ordinary and partial) lead to nonlinear integral equations. Of particular interest are the equations (1) $u(x)=\lambda\int_0^1K(x, y)f(y, u(y))dy$ (Hammerstein type) and (2) $u(x)=\lambda\int_0^1\Gamma(x, y, u(y))dy$ (generalized Hammerstein type). The author studies the problems: (I) to determine solutions of (1), (2) for small λ; (II) to continue the solutions obtained as an answer to (I). The emphasis is put on real solutions. First, analytic methods (successive approximations) are applied to (I); the results thus obtained are supplemented with the aid of fixed-point theorems [Banach, Caccioppoli, Schauder]. The latter approach to (1), (2) was used for the first time by V. V. Nemietzki. A typical result for (2) is as follows. Suppose that, for $\lambda=\lambda_0$, (2) has a solution $u(x)=u_0(x)$ and that $\Gamma(x, y, u_0(y)+z)=\sum_0^\infty A_\nu(x, y)z^\nu$ $(0\leqq x, y\leqq 1;\ |z|<R)$; if λ_0 is not a characteristic number of $A_1(x, y)$, then near $\lambda=\lambda_0$ (2) has a unique analytic solution, expressible in powers of $\lambda-\lambda_0$, such that $\lim u(x, \lambda)=u_0(x)$ as $\lambda\to\lambda_0$. It is indicated how to obtain an analytic continuation of the solution. Similar results hold for (1).

W. J. Trjitzinsky (Urbana, Ill.).

Functional Analysis

Dixmier, Jacques. Sur une classe nouvelle de variétés linéaires et d'opérateurs linéaires de l'espace de Hilbert. C. R. Acad. Sci. Paris **223**, 971–972 (1946).

The author defines a linear operator in Hilbert space to be of type d.e. if its graph (in the sense of von Neumann) is the domain of existence of some closed linear operator, and announces some properties of such operators. For ex-

ample, if A and B are of type d.e., so are $A+B$ and AB, and if A is of type d.e. then $A=ST^{-1}$, where S and T are bounded and everywhere defined. The closed linear operators, which of course form a subclass of the class of operators of type d.e., do not share the first two properties. The paper also contains a few remarks about properties of pairs of closed subspaces and their relationship to the graphs of various kinds of operators. *G. W. Mackey.*

Dixmier, Jacques. Propriétés géométriques des domaines d'existence des opérateurs linéaires fermés de l'espace de Hilbert. C. R. Acad. Sci. Paris **224**, 180–181 (1947).

Calling the domain of existence of a closed linear operator in Hilbert space a d.e. the author begins by investigating the family of closed subspaces contained in a d.e. and classifies d.e.'s according to the nature of this family. In the third section a class of sequences of positive real numbers is associated with each d.e. and it is asserted that two d.e.'s are unitary equivalent if and only if they are associated with the same class of sequences. The classes of sequences are such that a sequence is in the class belonging to some d.e. if and only if it is bounded away from zero, and two sequences $a_1, a_2, \cdots$ and $b_1, b_2, \cdots$ are in the same class if and only if there is a permutation $n_1, n_2, \cdots$ of the positive integers such that the sequences a_i/b_{n_i} and b_{n_i}/a_i are bounded. In the fourth section there is a discussion of the behavior of d.e.'s under intersection and linear union. In addition to describing how the classes to which $M\dotplus N$ and $M \cap N$ belong are related to the classes to which M and N belong the author announces a result characterizing d.e.'s as the subspaces one obtains by starting with the closed subspaces and iterating the operations of intersection and linear union a finite number of times. This last result was announced at about the same time by the reviewer [Bull. Amer. Math. Soc. **52**, 1009 (1946)]. In a concluding section the author asserts that his methods can be used to simplify the proofs of certain theorems of von Neumann.

G. W. Mackey (Cambridge, Mass.).

Dixmier, Jacques. Définition des opérateurs linéaires de l'espace d'Hilbert par leurs domaines d'existence et des valeurs. C. R. Acad. Sci. Paris **224**, 255–257 (1947).

For terminology see the two preceding reviews. Let D and D' be dense d.e.'s in Hilbert space. Then there always exists an operator of type d.e. with range D and domain D'. In this paper the author studies the problem of determining conditions on D and D' under which the operator may be chosen so as to be closed or have other special additional properties. For the case in which the special property is that of being bounded but not necessarily closed a theorem is stated which solves the problem completely. Among other results announced are the following. If neither D nor D' is closed and one fails to contain an infinite dimensional closed subspace then there exists no closed linear operator with range D and domain D'. There is to within unitary equivalence only one d.e. capable of being the domain of a closed symmetric operator with no self-adjoint extension.

G. W. Mackey (Cambridge, Mass.).

Ghezzo, S. Sulla conservazione dell'esistenza di radici di un sistema di equazioni, nel passaggio al limite. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **2**, 159–164 (1947).

The author establishes the following theorem. Let H denote Hilbert space, so that if $(x_1, x_2, \cdots)$ is in H, then $x_1^2+x_2^2+\cdots$ converges. Given the equations (1) $\phi_i(x_1, x_2, \cdots)=0$ $(i=1, 2, \cdots)$, with each ϕ_i continuous in H', a subset of H, and H'' a subset of H' which is compact with respect to H'. If, for each n, the system $\phi_i(x_1, x_2, \cdots)=0$ $(i=1, \cdots, n)$, has a root (a point) in H'', then (1) has a root in H'. As a corollary, the following theorem is established. Given the system (2) $\psi_i(x_1, x_2, \cdots)=0$ $(i=1, 2, \cdots)$, where the ψ_i's are continuous on the set C of points of H such that $-1\leqq x_i\leqq 1$ $(i=1, 2, \cdots)$, suppose that (on C)

$$\psi_i(x_1, \cdots, x_{i-1}, -1, x_{i+1}, \cdots)\leqq 0,$$
$$\psi_i(x_1, \cdots, x_{i-1}, 1, x_{i+1}, \cdots)\geqq 0$$

and that there is a compact set I of H such that, for each n, the solutions of the system $\psi_i(x_1, \cdots, x_n, 0, 0, \cdots)=0$ $(i=1, \cdots, n)$ are contained in I. Then (2) has a solution in H. The following theorem is a further corollary. If $Y=\Gamma(X)$ is a transformation of all points X of C into points Y of a subset H^* of H, such that Γ is continuous, and such that, for each i $(i=1, 2, \cdots)$, the sets of points of C for which $x_i=-1$ and for which $x_i=1$ are carried, respectively, either into sets for which $x_i\leqq -1$ and $x_i\geqq 1$, or into sets for which $x_i\geqq -1$ and $x_i\leqq 1$, then the transformation has a fixed point. In particular, the transformation has a fixed point if it is continuous and carries C into a compact subset of H, and carries the boundary of C into a subset of C.

A. B. Brown (Flushing, N. Y.).

Mackey, George W. On convex topological linear spaces. Trans. Amer. Math. Soc. **60**, 519–537 (1946).

L'auteur développe les résultats annoncés précédemment [Proc. Nat. Acad. Sci. U. S. A. **29**, 315–319 (1943); **30**, 24 (1944); ces Rev. **5**, 99], concernant les relations entre les topologies d'espace localement convexe définies sur un espace vectoriel X, et l'espace vectoriel L des formes linéaires continues sur X pour une telle topologie. En dehors des résultats énumérés dans le compte-rendu du travail précité [il faut rectifier dans ce compte-rendu l'énoncé de la condition pour qu'un espace "relativement faible" soit métrisable: il faut et il suffit qu'il soit isomorphe à un sous-espace de l'espace (s) de Banach, et non à cet espace lui-même], l'auteur caractérise les espaces X tels que toute application linéaire de X dans un espace localement convexe, qui transforme les ensembles bornés en ensembles bornés, soit continue. Dans l'étude des diverses définitions d' "espace complet" données pour les espaces localement convexes, il montre entre autres comment on peut toujours "compléter partiellement" X de sorte que les filtres de Cauchy sur X qui contiennent un ensemble borné, soient convergents dans l'espace ainsi complété. [Le rapporteur signale que dans les lignes 27–28 de la p. 523, les mots "stronger" et "weaker" doivent être intervertis.] *J. Dieudonné* (Nancy).

Katětov, Miroslav. Zur Theorie der topologischen Vektorräume. Acad. Tchèque Sci. Bull. Int. Cl. Sci. Math. Nat. **44**, 599–605 (1943).

The author studies convex topological linear spaces and the more general objects obtained by dropping the requirement that the topology separate points. In the first section he presents some results on the connection between convex sets and continuous linear functionals and on linear mappings which have continuity properties with respect to the weak topologies. The second section is devoted to a discussion of a relationship of duality between convex topological linear spaces, two such spaces being dual when there is a bilinear functional defined on their direct product which in the obvious fashion sets up a one-to-one mapping of each

space onto the space of continuous linear functionals of the other. Every convex topological linear space has a dual which is not in general unique. Among other things it is proved that if two dual spaces are normable and one is of the second category then they are both reflexive and each is isomorphic to the conjugate of the other. In the last section the relative strength of various convex topologies in the same space is considered and it is shown, in particular, that there is always a strongest and a weakest such topology in the family of all of those "L equivalent" to a given one. Here two topologies are said to be L equivalent if they render the same linear subspaces closed. As the author remarks, some of his results are more or less contained in a paper of Dieudonné [Ann. Sci. École Norm. Sup. (3) **59**, 107–139 (1942); these Rev. **6**, 178] which apparently came to his attention after he had obtained the results but before he had published them. Some of them were also obtained independently at about the same time by the reviewer [cf. the preceding review]. *G. W. Mackey.*

Macphail, M. S. Absolute and unconditional convergence. Bull. Amer. Math. Soc. **53**, 121–123 (1947).

L'auteur montre que, pour que toute série commutativement convergente dans un espace de Banach B soit absolument convergente, il faut et il suffit qu'il existe un nombre $a>0$ tel que, pour toute famille finie (x_i), $i\epsilon I$, de points de B, on ait

$$\sum_{i\epsilon I}\|x_i\|\leqq a \sup_{J\subset I}\|\sum_{i\epsilon J}x_i\|$$

Il prouve que cette condition n'est pas vérifiée dans les espaces L et l. *J. Dieudonné* (Nancy).

Segal, I. E. Irreducible representations of operator algebras. Bull. Amer. Math. Soc. **53**, 73–88 (1947).

Par une structure de (*)-algèbre, on entend la structure qui comporte la donnée d'une structure d'algèbre sur le corps C des complexes, et d'un antiautomorphisme involutif $x\rightarrow x^*$ de cette algèbre, tel que $(\alpha x)^*=\bar{\alpha}x^*$ pour $\alpha\epsilon C$. Une fonction linéaire $F(x)$, à valeurs dans C, sur une (*)-algèbre A, pourra être dite de type positif si $F(x^*)=\overline{F(x)}$ et $F(x^*x)\geqq 0$ quel que soit $x\epsilon A$; une telle fonction détermine sur A le produit scalaire $(x, y)=F(x^*y)$; par passage au quotient A/B de A par l'ensemble B des $z\epsilon A$ tels que $F(z^*z)=0$, et complétion, on obtient un espace de Hilbert H; d'après l'inégalité de Schwarz, B est aussi l'ensemble des $z\epsilon A$ tels que $F(uz)=0$ quel que soit $u\epsilon A$, et est donc idéal à gauche; par suite A est anneau d'opérateurs (à gauche) sur A/B; si on admet l'existence sur A d'une norme $\|x\|$ soumise à des hypothèses convenables (e.g., s'il existe dans A un élément unité e, et si $\|x\|\leqq 1$ entraîne qu'il existe $y\epsilon A$ tel que $x^*x+y^*y=e$), on aura $(ux, ux)\leqq\|u\|^2\cdot(x, x)$, et l'opérateur sur A/B déterminé par $u\epsilon A$ pourra être prolongé à un opérateur borné T_u sur H; on a ainsi une représentation $u\rightarrow T_u$ de la (*)-algèbre A par une (*)-algèbre (autoadjointe) d'opérateurs bornés dans H. Les notions précédentes sont à la base du travail de Gelfand-Neumark [Rec. Math. [Mat. Sbornik] N.S. **12**(54), 197–213 (1943); ces Rev. **5**, 147]. Si A est l'algèbre A_G d'un groupe G localement compact (définie par le produit de composition, soit sur l'espace L^1 formé sur G au moyen de la mesure de Haar, soit encore sur l'ensemble des mesures de Radon sur G), les fonctions de type positif sur A_G sont étroitement apparentées aux fonctions de type positif sur le groupe G, et on est conduit ainsi aux méthodes introduites par Gelfand-Raikov [Rec. Math. [Mat. Sbornik] N.S. **13**(55), 301–316 (1943); ces Rev. **6**, 147], et, indépendamment, par Godement [C. R. Acad. Sci. Paris **221**, 69–71, 134–136, 686–687 (1945); **222**, 36–37, 213–215 (1946); ces Rev. **7**, 254, 255, 241, 454] dans l'étude des représentations unitaires de G.

Ces diverses idées sont reprises et développées par l'auteur, en supposant A donnée comme (*)-algèbre (autoadjointe) d'opérateurs bornés X dans un espace de Hilbert H_0; pour norme $\|X\|$ il prend la borne, sup $\|Xz\|/\|z\|$; il suppose A complète pour $\|X\|$, ce qui entraîne la possibilité, pour $X=X^*$, de définir $f(X)$ quelle que soit la fonction continue numérique $f(t)$ de la variable réelle t, s'annulant pour $t=0$; si $|f(t)|\leqq 1$ sur un intervalle contenant le spectre de X, on a $\|f(X)\|\leqq 1$. Pour les fonctions linéaires continues $F(X)$ sur A, à valeurs dans C, on peut définir une norme $\|F\|=\sup|F(X)|/\|X\|$; si F et F' sont de type positif sur A, on peut montrer qu'on a $\|F+F'\|=\|F\|+\|F'\|$; en particulier, si A contient l'opérateur identique I, on a $\|F\|=F(I)$ pour F de type positif. L'ensemble des F de type positif, telles que $\|F\|\leqq 1$, est convexe, et on peut lui appliquer le théorème de Krein-Milman, qui garantit l'existence de points extrémaux; ces derniers sont qualifiés par l'auteur d'"états purs" de A, un "état" de A étant une fonction F de type positif telle que $\|F\|=1$ (ces dénominations sont d'origine quantique). De là résulte l'existence d'un système complet de représentations irréductibles pour A, au moyen du théorème suivant, le principal du présent travail: tout "état" de A peut être écrit, d'une manière et essentiellement d'une seule, sous la forme $F(X)=(\Phi(X)a, a)$, où $X\rightarrow\Phi(X)$ est une représentation de A par une (*)-algèbre d'opérateurs bornés dans un espace de Hilbert H, et où a est un vecteur de H tel que $\|a\|=1$ et que les $\Phi(X)a$, pour $X\epsilon A$, soient partout denses dans H; réciproquement, si Φ et a sont tels, $(\Phi(X)a, a)$ est un "état" de A; pour que cet "état" soit "pur," il faut et il suffit que Φ soit irréductible.

En particulier, soit φ une représentation (évidemment irréductible) de A dans un espace H à une dimension, c'est-à-dire une fonction linéaire $\varphi(X)$ sur A, à valeurs dans C, telle que $\varphi(X^*)=\overline{\varphi(X)}$ et que $\varphi(XY)=\varphi(X)\varphi(Y)$; il s'ensuit qu'une telle fonction (que l'auteur propose d'appeler une "observation" de A) est un "état pur." L'auteur montre (au moyen du théorème de Hahn-Banach) que toute "observation" $\varphi(X)$ d'une sous-(*)-algèbre A_1 de A peut être prolongée à un "état pur" de A, et en déduit l'existence d'une représentation irréductible $\Phi(X)$ de A, et d'un vecteur a dans l'espace de Hilbert où opère celle-ci, tels que l'on ait, pour $X\epsilon A_1$, $\Phi(X)a=\varphi(X)a$. L'auteur applique d'autre part son résultat principal au cas de l'algèbre d'un groupe localement compact. *A. Weil* (Chicago, Ill.).

Romanoff, N. P. On one-parameter groups of linear transformations. I. Ann. of Math. (2) **48**, 216–233 (1947).

This is a translation of the first half of a Russian paper with the same title issued as a mimeographed communication from the Siberian Physico-Technical Institute of the University of Tomsk in 1942. The author is concerned with one-parameter groups (occasionally semi-groups) of linear transformations L_u with the composition property $L_uL_v=L_{uv}$. The parameter manifold is either the set of all positive numbers ($u\geqq 1$ in the semi-group case) or the set of all complex numbers $u\neq 0$; L_u operates in a function space F of elements $f(x)$ defined for all x in a domain D. No topology is introduced in F. It is supposed that $L_u(F)\subset F$ and L_1 is the identity transformation. The decisive assumption is that

L_u is differentiable at $u=1$, that is, $\lambda f=[(\partial/\partial u)L_uf(x)]_{u=1}$ exists for all $f(x)\epsilon F$, so λ is a linear operator defined on F. This set of postulates is denoted by S_1 in the real and S_1^* in the complex case. If $\lambda(F)\subset F$ and $\lambda L_u=L_u\lambda$, we get the postulates S_2 and S_2^*. Formally $L_u=u^\lambda$. The greater part of the paper is devoted to a description of eight different methods of constructing such groups. Most of the formulas being quite complicated, brief indications of the methods will have to suffice here.

Method A: $L_uf=u^\lambda f=\exp(\lambda\log u)f$, where the exponential function is defined by the power series; this presupposes postulates S_3, but if the power series can be summed in closed form the latter may have a wider range of validity. Method B: If L_u is a group acting in F and if the mapping $F^*=\alpha F$ is one-to-one, then $L_u^*=\alpha L_u\alpha^{-1}$ is a one-parameter group in F^*. Method C: Set $L_u^*=\int_0^\infty K(u,v)L_vfdv$, where $\int_0^\infty K(u_1,v)K(u_2,w/v)dv/v=K(u_1u_2,w)$. Method D: Use of Laplace transforms. If $\int_0^\infty\varphi(x,u)e^{-sx}dx$ is a power of u, the exponent of which is a function of s alone, set $L_uf=\int_0^xf(t)\varphi(x-t,u)dt$. Method E: If A_u and B_v are commuting semi-groups, then $C_u=A_uB_u$ is a one-parameter semi-group. Method F: One can pass from the operator $\exp(\lambda\log u)$ to $\exp(\lambda^n\log u)$ by an integral involving a kernel $Y_n(t)$ which satisfies the differential equation $y^{(n-1)}(t)=(-1)^{n-1}ty(t)$. Method G: If A_u is a differentiable one-parameter Lie group in R_n, $A_uA_v=A_{uv}$, then $L_uf(P)=f(A_uP)$, $P=(x_1,\cdots,x_n)$, is a one-parameter group. Method H: If $\{\psi_n(x)\}$ is an orthonormal system for the interval I, set

$$L_uf=\sum u^{\lambda_n}\psi_n(x)\int_I\psi_n(t)f(t)dt$$

with suitable choice of the constants λ_n.

The operand spaces are determined with care; they are frequently quite restricted since the author usually wants to obtain a full group rather than a semi-group. All methods are illustrated by a profusion of special important cases. No attempt is made to derive the differentiability properties from assumptions of continuity or measurability. The second half of the paper will deal with operational gamma and zeta functions. *E. Hille* (New Haven, Conn.).

Calculus of Variations

Mancill, Julian D. Unilateral variations with variable end-points. Amer. J. Math. **69**, 121–138 (1947).

The author first derives a transversality condition which must be satisfied by the endpoints of a plane arc E_{12} which minimizes an integral of the form

$$I=\int_{t_1}^{t_2}F(x,y,x',y')dt$$

in a class of admissible arcs joining two fixed curves C and D when it is assumed that the arc E_{12} lies on the boundary of the region of admissible arcs. This transversality condition involves an inequality instead of the equality which one obtains with free instead of unilateral variations. He then establishes a sufficiency theorem when E_{12} nowhere satisfies the Euler equations. Finally, when E_{12} is a solution of the Euler equations he states a group of sufficiency theorems, differing from each other in their assumptions concerning the Jacobi condition. *J. E. Wilkins, Jr.*

Hestenes, Magnus R. An alternate sufficiency proof for the normal problem of Bolza. Trans. Amer. Math. Soc. 61, 256–264 (1947).

The author proves two main results relative to sufficiency theorems for the problem of Bolza. He essentially shows that the sufficiency theorems for strong or weak relative minima for a normal problem can be got from the corresponding results for a related problem having no differential side conditions. He considers a functional $I(C)=g(a)+\int_Cf(a,y,\dot y)dt$ over a class of arcs a^h, $y^i(t)$ which satisfy the end conditions $y^i(t^1)=T^{i1}(a)$, $y^i(t^2)=T^{i2}(a)$ and the side conditions $\varphi^\beta(a,y,\dot y)=0$. It is shown that to an arc C_0, satisfying the usual sufficiency conditions for this problem, there corresponds a set $m^\beta(a,y,\dot y)$ of functions such that C_0 satisfies the corresponding sufficiency conditions with respect to the functional

$$J(C)=g(a)+\int_C\{f(a,y,\dot y)+\sum m^\beta(a,y,\dot y)\varphi^\beta(a,y,\dot y)\}dt$$

in the class of arcs satisfying merely the end conditions. It then follows that $J(C_0)$ is a strong (weak) relative minimum in the class of arcs satisfying the end conditions. A fortiori it is a minimum in the smaller class of those arcs which also satisfy the side conditions; but for those arcs $J(C)$ is evidently $I(C)$. Simple explicit formulas are given for the m^β in the course of the proofs. *H. H. Goldstine.*

Sakellariou, Neilos. On some problems of the calculus of variations. Bull. Soc. Math. Grèce **22**, 155–167 (1946). (Greek)

***Cinquini-Cibrario, Maria. Relazioni fra integrali doppi e soluzioni di equazioni a derivate parziali.** Atti Secondo Congresso Un. Mat. Ital., Bologna, 1940, pp. 112–118. Edizioni Cremonense, Rome, 1942.

on order

This paper is concerned with a double integral problem of the calculus of variations,

$$I(z)=\int\int_Df(x,y,z,z_x,z_y)dxdy,$$

in which the integrand function $f(x,y,z,p,q)$ satisfies the following conditions for (x,y) in the open domain D and z,p,q arbitrary finite values: (i) $f(x,y,z,p,q)$ is of class C'' and nonnegative; (ii) the ternary quadratic form whose coefficients are the second order partial derivatives of f with respect to z,p,q is positive semidefinite; (iii) for each $H>0$ there is a positive constant M such that when z is restricted by $|z|\leqq H$ we have $|f_z(x,y,z,p,q)|\leqq M$. Under these hypotheses it is shown that, if $z_0(x,y)$ is continuous on the closure $\bar D$ of D and is of class C'' and a solution of the Euler equation in D, while there exists a function $z(x,y)$ which is continuous on $\bar D$, coincides with $z_0(x,y)$ on $\bar D-D$, is absolutely continuous in the sense of Tonelli on D, and such that $I(z)$ is finite, then $I(z_0)$ is finite also and $I(z_0)\leqq I(z)$. Conditions are discussed under which $I(z_0)<I(z)$ unless $z\equiv z_0$, and applications of the result cited above are noted. The author was evidently unaware of a paper by E. J. McShane [Bull. Amer. Math. Soc. **40**, 593–598 (1934)] in which a general treatment of the minimizing property of a harmonic function is given; indeed, for the special case of the Dirichlet integral the results of the present paper are weaker and less complete than those of McShane. *W. T. Reid* (Evanston, Ill.).

van Hove, Léon. Sur l'extension de la condition de Legendre du calcul des variations aux intégrales multiples à plusieurs fonctions inconnues. Nederl. Akad. Wetensch., Proc. **50**, 18–23 = Indagationes Math. **9**, 3–8 (1947).

Consider the integral

$$I(V)=\int_V\cdots\int F(x_1,\cdots,x_m;y^1,\cdots,y^n;\overset{1}{p}_1,\cdots,\overset{n}{p}_m)dx_1\cdots dx_m$$

in which V is a variety defined by functions $y^i(x)$ and $p_\alpha^i=\partial y^i/\partial x_\alpha$. It is known that, if V_0 is an extremal which minimizes $I(V)$ in the class of varieties having the same boundary as V_0, then the Legendre form $Q=F_{p_\alpha^i p_\beta^k}\pi_\alpha^i\pi_\beta^k$ must be nonnegative whenever the matrix $\pi=\|\pi_\alpha\|$ has rank one. If either $n\leqq 2$ or $m\leqq 2$, then Terpstra [Math. Ann. **116**, 166–180 (1938)] and Hestenes and MacShane [Trans. Amer. Math. Soc. **47**, 501–512 (1940); these Rev. **2**, 119] have shown that, if $Q>0$ at some point P on an extremal V_0 whenever π has rank one, then V_0 furnishes a weak local minimum for $I(V)$, i.e., $I(V)>I(V_0)$ whenever V is a variety which has the same boundary as V_0, lies near V_0 in (x, y, p)-space, and differs from V_0 only in a neighborhood of the point P. By the use of Fourier transforms the author shows that this result remains true without the restriction that n or m does not exceed two.

J. E. Wilkins, Jr. (Buffalo, N. Y.).

Theory of Probability

Hoyt, Ray S. Probability functions for the modulus and angle of the normal complex variate. Bell System Tech. J. **26**, 318–359 (1947).

Let $W=U+iV$, where $i^2=-1$ and U and V are independent normal variates with mean 0 and variances s_u^2 and s_v^2. Let $s^2=s_u^2+s_v^2$ and $b=(s_u^2-s_v^2)/s^2$. Finally let $W=Re^{i\theta}$, where R and θ are real. The probability density of R can be expressed in terms of Bessel functions. Curves are given corresponding to $s=1$ and $b=[0(.1)1]$ for $0\leqq R\leqq 2.8$ and larger graphs for $0\leqq R\leqq .4$ corresponding to the same b-values and also $b=.95, .98, .99, .995, .999, .9999$. As $b\to 1$ the curves converge but not uniformly. A sort of Gibbs phenomenon is clearly visible. The densities for R^{-1} and for θ and the corresponding cumulative distribution functions are computed and graphed in a similar way. The best methods for numerical computation are discussed.

W. Feller (Ithaca, N. Y.).

Théodoresco, N. Un problème de loterie. Disquisit. Math. Phys. **1**, 339–356 (1941).

An urn contains n black and b white balls. Drawings are made without replacements until m black balls are extracted. Then b white balls are added and a new series of drawings is made according to the same rule. The procedure is repeated r times or until all black balls are extracted. The expected numbers of white balls extracted during the last period and of white balls remaining in the urn are calculated. Applications are made to the Romanian government lottery.

W. Feller (Ithaca, N. Y.).

Hostinsky, B. Probabilités relatives aux tirages de deux urnes avec l'échange des boules extraites. Acta [Trudy] Univ. Asiae Mediae. Ser. V-a. Fasc. **21**, 10 pp. (1939). (French. Russian summary)

Consider the Markov chain defined by the following urn problem. A total of N black and N white balls is distributed in two urns A and B. From each urn a ball is chosen at random and the two balls are interchanged. The author computes the stationary probabilities and the characteristic equation for $N=2, 3, \cdots, 6$. *W. Feller* (Ithaca, N. Y.).

Sarymsakov, T. A. The law of the iterated logarithm for Markov schemes. Acta [Trudy] Univ. Asiae Mediae. N. S. Math. Fasc. **5**, 15 pp. (1945). (Russian)

For a Markov chain with two possible states the author proves that the number of passages through either state obeys the law of the iterated logarithm. [For the analogous result for arbitrary finite chains cf. Doeblin, Sur les propriétés asymptotiques de mouvements régis par certains types de chaînes simples, thèse, Paris, 1938.]

W. Feller (Ithaca, N. Y.).

Romanovskiĭ, V. I. On the probabilities of the recurrence of cycles in polycyclic chains. Acta [Trudy] Univ. Asiae Mediae, N. S. Fasc. **7**, 20 pp. (1946). (Russian)

This paper discusses problems involving a Markov chain of the following type. Let $A_1, \cdots, A_n$ denote the possible states of a system. Let $B_i=(A_{1+t_{(i-1)}}, A_{2+t_{(i-1)}}, \cdots, A_{t_i})$, $i=1, 2, \cdots, m$; $t_m=n$. The initial probabilities are zero for all states except those which are members of B_1. From B_1 the system can move to B_2 or B_{a+1}, with prescribed transition probabilities. If the system has moved to B_2 it must then move through $B_3, B_4, \cdots, B_a$, back to B_1; in this case the system is said to have passed through cycle C_1. If from B_1 the system has moved to B_{a+1}, it must then move through $B_{a+2}, B_{a+3}, \cdots, B_m$ back to B_1, and is said to have passed through cycle C_2. The description above is that of a bicyclic chain; the generalization to polycyclic chains is immediate.

The author concerns himself chiefly with bicyclic chains. He studies such problems as finding the probability that, in passing through $k+l$ cycles, the system will pass k times through cycle C_1. Transition probabilities, moment generating functions and asymptotic distributions are obtained.

J. Wolfowitz (New York, N. Y.).

Kac, Mark, and Siegert, A. J. F. On the theory of noise in radio receivers with square law detectors. J. Appl. Phys. **18**, 383–397 (1947).

Les auteurs calculent la densité de probabilité $P(V)$ de la tension V en bruit de fond à la sortie d'un dispositif du type suivant: la source de bruit applique la tension en bruit entrante à un amplificateur de fréquence intermédiaire A.F.I. de fréquence passante f_0, de fonction de réponse $Q(\tau)$ (avec $Q(\tau)=0$ pour $\tau<0$), lequel applique sa tension sortante à un amplificateur video (ou audio) A.V. à fonction de réponse $K(\tau)$ ($K(\tau)=0$ pour $\tau<0$), mais entre A.F.I. et A.V. est intercalé un détecteur quadratique; on pose $Q(\tau)=\int_{-\infty}^{\infty}e^{2\pi i(f-f_0)\tau}\gamma(f)df$, en supposant $\int_{-\infty}^{\infty}|\gamma^2(f)|df=1$ et $\gamma(f_0+f)=\gamma^*(f_0-f)$; la largeur de la bande passante de A.F.I. est supposée faible par rapport à f_0; on suppose $\int_{-\infty}^{\infty}K(\tau)d\tau=1$ et on admet avec Rice [cf. Bell System Tech. J. **23**, 282–332 (1944); ces Rev. **6**, 89] que la tension en bruit de fond sortant de A.F.I. est de la forme $X(t)\cos 2\pi f_0t+Y(t)\sin 2\pi f_0t$, où

$$X(t)=(N/T)^{\frac{1}{2}}\sum_{-\infty}^{\infty}|\gamma(f_0+f_k)|(X_k\cos 2\pi f_kt+Y\sin 2\pi f_kt),$$

$$Y(t)=(N/T)^{\frac{1}{2}}\sum_{-\infty}^{\infty}|\gamma(f_0+f_k)|(-X_k\sin 2\pi f_kt+Y_k\cos 2\pi f_kt),$$

T étant un intervalle de temps determiné long, N la puis-

sance en bruit de fond entrante, les X_k et les Y_k des variables aléatoires indépendantes de même densité $\pi^{-\frac{1}{2}}e^{-u^2}$.

Dans le cas d'un bruit pur (pas de signal), les auteurs expriment $P(V)$ en fonction des valeurs propres de l'équation intégrale

$$\int_0^\infty K(\theta)\rho(\theta'-\theta)f(\theta)d\theta=\lambda f(\theta'),$$

où

$$\rho(\theta'-\theta)=\int_{-\infty}^{\infty}Q(\tau)Q(\tau+\theta'-\theta)d\tau;$$

dans le cas mixte (bruit et signal), $P(V)$ est de même exprimée à l'aide des valeurs propres d'une équation intégrale (par l'intermédiaire de sa caractéristique). Les calculs sont poussés à bout dans des cas particuliers où les valeurs propres des équations intégrales sont déterminées explicitement; une expression approchée de $P(V)$ est donnée dans le cas où la largeur de bande de A.V. est grande par rapport à celle de A.F.I. *R. Fortet* (Caen).

Cameron, R. H., and Martin, W. T. The orthogonal development of non-linear functionals in series of Fourier-Hermite functionals. Ann. of Math. (2) **48**, 385–392 (1947).

The authors define a complete orthonormal set of functions on the space of continuous functions with the usual Wiener Brownian movement measure. *J. L. Doob.*

Cameron, R. H., and Martin, W. T. Fourier-Wiener transforms of functionals belonging to L_2 over the space C. Duke Math. J. **14**, 99–107 (1947).

The authors modify slightly their previous definition of a Fourier-Wiener transform defined on functions in L_2 of the space in the preceding review [Duke Math. J. **12**, 485–488, 489–507 (1945); these Rev. **7**, 62] and find a simple expression for the transform in terms of the complete orthonormal set defined in the paper reviewed above. *J. L. Doob.*

Jost, Res. Bemerkungen zur mathematischen Theorie der Zähler. Helvetica Phys. Acta **20**, 173–182 (1947).

It is shown how the theory of summation of random variables and the method of Laplace transforms lead in a simple way to the mean and variance of the number of registrations in a counter or in two counters in series. *W. Feller* (Ithaca, N. Y.).

Lovera, Giuseppe. Sullo scarto quadratico medio nei conteggi con i contatori. Ricerca Sci. **17**, 223–228 (1947).

Simple numerical approximations to the mean and variance of the number of registrations in a counter obtained by Ruark and Devol [Physical Rev. (2) **49**, 355–367 (1936)]. Comparison with experiments. *W. Feller.*

Mathematical Statistics

Andreoli, Giulio. Statistica degli aggregati in una collettività e concentrazione rispetto a due caratteri. Rend. Accad. Sci. Fis. Mat. Napoli (4) **10**, 160–172 (1940).

The author distinguishes between cumulative characters (like mass or wealth) and noncumulative ones (e.g., temperature). He describes some additive properties for the composition of two populations. *W. Feller.*

Andreoli, Giulio. Statistica di configurazioni. (Ricerche su coppie di variabili casuali in correlazione). Rend. Accad. Sci. Fis. Mat. Napoli (4) **11**, 150–158 (1941).

In vague terms the author suggests that the joint distribution of two random variables with finite ranges is a matrix and that the evolution of a system is described by a succession of such matrices which might be analyzed. *W. Feller* (Ithaca, N. Y.).

Martin, D. On the radial error in a Gaussian elliptical scatter. Philos. Mag. (7) **37**, 636–639 (1946).

Let X and Y be uncorrelated normal variables with variances h^2 and k^2. It is shown that the probability of $X^2+Y^2<r^2$ equals

$$\exp\{-r^2(h^2+k^2)/2\}\sum(a^n-a^{-n})I_n\{r^2(k^2-h^2)/2\},$$

where $a=(k+h)/(k-h)$. The mean of $(X^2+Y^2)^{\frac{1}{2}}$ is expressed by an elliptic integral. *W. Feller* (Ithaca, N. Y.).

Sakamoto, H. On the distributions of the product and the quotient of the independent and uniformly distributed random variables. Tôhoku Math. J. **49**, 243–260 (1943).

Suppose $X_1,\cdots,X_n$ are nonnegative random variables and uniformly distributed on finite intervals. Noting that $\prod_{i=1}^n X_i=\exp\{\sum_{i=1}^n \log X_i\}$, the author, by making an application of the Fourier transform, finds an expression for the exact distribution of $\sum_{i=1}^n \log X_i$ and then of $\prod_{i=1}^n X_i$. Similarly, he finds an expression for the exact distribution of the ratio of two random variables X_1/X_2. *S. S. Wilks* (Princeton, N. J.).

Dodd, E. Some internal and external means arising from the location of frequency distributions. Acta [Trudy] Univ. Asiae Mediae. Ser. V-a. Fasc. **23**, 8 pp. (1939). (English. Russian summary)

This is one of a series of articles by the author on the classification of means. We are given a sample of n measurements $\{x_i\}$, each with density function $f(x_i)$. Let $\xi_i=(x_i-m)/a$, where m is some location constant and a a scalar constant. Fisher's likelihood function is $L=C\prod_{i=1}^n f(\xi_i)$. The solution of the equation $\partial L/\partial m=0$ gives an $m=F\{x_i\}$ which is the so-called substitutive mean; m may be multiple-valued, of course. In general, L is maximized (if at all) by an internal mean. However, often with a bimodal (or multimodal) distribution, only an external mean (m less or greater than all x_i) will maximize L, while the internal mean (or means) minimizes L. *R. L. Anderson* (Raleigh, N. C.).

Kolmogorov, A. N. On the proof of the method of least squares. Uspehi Matem. Nauk (N.S.) **1**(11), no. 1, 57–70 (1946). (Russian)

The author criticizes general textbook expositions of the method of least squares on two counts: they fail to indicate that the Gaussian error law seriously overestimates the reliability of the results derived from small samples and they derive their main results by a cumbersome set of calculations rather than by the lucid methods of vector algebra. The paper is written to show how this condition can be corrected.

The vector methods are illustrated as follows. Let $y, x_1, \cdots, x_n$ satisfy a linear relation $y=\sum_{j=1}^n a_j x_j$, the a_j being unknown constants. We make N experimental observations on the y and x's, thus determining a set of $n+1$ vectors in Euclidean N-space, with components η_r, ξ_{jr}, $r=1,\cdots,N\geqq n$. We suppose that the rank of the matrix $\|x_{jr}\|$ is n. The linear vector equation $\eta=\sum_{j=1}^n a_j\xi_j$ cannot in general be satisfied; we seek, therefore, the most reason-

able set of values α_j to approximate the a_j. Write $\eta=y+\Delta$, $\eta^*=\sum_{j=1}^n\alpha_j\xi_j$ and $\epsilon=\eta-\eta^*$. It is clear that η^* belongs to the linear subspace L spanned by the x_j. Denoting scalar products by [ ], we see that the condition $[\epsilon\epsilon]$=minimum is equivalent to the condition that η^* is the orthogonal projection of η on L, whence $[\epsilon\xi_j]=0$ follows. A further immediate consequence is that $\sum_{j=1}^n[\xi_i\xi_j]\alpha_j=[\xi_i\eta]$, $i=1,\cdots,n$. These are the normal equations for the α_j and have a solution since determinant $[\xi_i\xi_j]\neq 0$.

Next, define a set of vectors $u_i\epsilon L$ by $[u_i\xi_j]=\delta_{ij}$ (Kronecker symbols) and write $[u_iu_j]=q_{ij}$. Then $\alpha_j=a_j+[\Delta u_j]$. If we suppose that the components Δ_r of Δ are random variables with $M\Delta_r=0$, $M\Delta_r\Delta_t=\delta_{rt}s^2$ (with s finite and independent of r,t and with M the appropriate mean value operator), we find that $M\alpha_j=a_j$ and $M(\alpha_i-a_i)(\alpha_j-a_j)=q_{ij}s^2$. Similarly, one derives $M\epsilon=0$ and $M[\epsilon\epsilon]=(N-n)s^2$.

The χ^2-distribution and Student's distribution are derived and there are brief discussions of confidence limits and of the significance of the dispersion matrix q_{ij}.

A. A. Brown (Alexandria, Va.).

Tweedie, M. C. K. The regression of the sample variance on the sample mean. J. London Math. Soc. 21, 22–28 (1946).

Define $M(t)=E[\exp(-tx)]$, $U_n(t)=E[s^2\exp(-t\sum x_i)]$, where $s^2=\sum_1^n(x_i-\bar{x})^2/(n-1)$ is the usual estimate for the variance from a sample $x_1,\cdots,x_n$ of independent observations. It is shown that (1) $U_n=M^nd^2(\log M)/dt^2$. By inversion of U_n and M^n it is possible to compute $v_n(\bar{x})$, the conditional expectation of s^2 for fixed $\bar{x}$. If v_n is a given polynomial in $\bar{x}$, (1) reduces to a differential equation in M, from which the distribution of x can be computed. The distributions are computed for polynomials of degree 0, 1 and 2; if v_n is constant, for instance, the author obtains the known result that x is Gaussian.

D. Blackwell.

Neyman, J. On one fundamental problem of the mathematical statistics. Acta [Trudy] Univ. Asiae Mediae. Ser. V-a. Fasc. 29, 12 pp. (1939). (Russian. English summary)

An exposition in brief form of the Neyman-Pearson theory of testing hypotheses. The major references are to J. Neyman and E. S. Pearson, Philos. Trans. Roy. Soc. London. Ser. A. 231, 289–337 (1933); Statistical Research Mem. 1, 113–137 (1936); J. Neyman, Bull. Soc. Math. France 63, 246–266 (1935).

A. A. Brown (Alexandria, Va.).

Bhattacharyya, A. On some analogues of the amount of information and their use in statistical estimation. Sankhyā 8, 1–14 (1946).

The main result is a generalization of an inequality found by Cramér [Mathematical Methods of Statistics, Uppsala, 1945; Princeton University Press, 1946; these Rev. 8, 39] and by Rao [Bull. Calcutta Math. Soc. 37, 81–91 (1945); these Rev. 7, 464], giving a lower bound for the variance of any estimate of a parameter. Suppose the sample $x=(x_1,\cdots,x_n)$ has the probability density $f(x;\theta)$, where θ is a parameter. Define $\varphi^{(i)}(x;\theta)=\partial^if/\partial\theta^i$, and let $T=T(x)$ be a statistic. Calculate the expected values $J_{ij}(\theta)=E(\varphi^{(i)}\varphi^{(j)})$ and $\tau(\theta)=E(T)$ with the density $f(x;\theta)$. Let $\tau^{(i)}(\theta)=\partial^i\tau/\partial\theta^i$. If ν is such that the $\nu\times\nu$ matrix (J_{ij}) is nonsingular, where $i,j=1,\cdots,\nu$, so that its reciprocal (J_ν^{ij}) exists, then the result is

$$(*)\qquad \mathrm{Var}\,(T)\geqq\sum_{i,j=1}^{\nu}J_\nu^{ij}\tau^{(i)}\tau^{(j)}.$$

The mathematical restrictions under which the results are valid are not all stated; in the proof of (*) certain improper integrals are differentiated under the integral sign.

H. Scheffé (Los Angeles, Calif.).

Bowker, Albert H. Computation of factors for tolerance limits on a normal distribution when the sample is large. Ann. Math. Statistics 17, 238–240 (1946).

A simple asymptotic formula is given for the factor λ proposed by Wald and Wolfowitz [same Ann. 17, 208–215 (1946); these Rev. 8, 478] for tolerance limits $x\pm\lambda s$ for a normal distribution. A table is given, comparing values of λ from the formula with the exact values.

D. Blackwell.

Derevitsky, N. On the rejection of field experiment data and their subsequent analysis. Acta [Trudy] Univ. Asiae Mediae. Ser. V-a. Fasc. 22, 21 pp. (1939). (Russian. English summary)

The author discusses Chauvenet's principle (from a sample of N observations, reject those less probable than $1/2N$) as a guide to the detection of gross errors in field experiments. Having rejected certain items from an mn-fold experiment, he determines the theoretical values of the missing observations by least squares [Allan and Wishart, J. Agricultural Sci. 20, 399–406 (1930)]. He offers the approximation process

$$a_{ij}(p+1)=\frac{nS_i(p)+mS_j(p)-S_t(p)-(m+n-1)a_{ij}(p)}{(m-1)(n-1)}$$

as a rapid method of calculating the desired theoretical values a_{ij}. Here $S_i(p)$, $S_j(p)$ and $S_t(p)$ are the computed mean yields for the ith treatment, jth replication and total experiment, respectively, using the observed values of accepted data and the pth order approximations $a_{ij}(p)$ as values of the missing data; there are m treatments and n replications. Several numerical examples are discussed at length.

A. A. Brown (Alexandria, Va.).

TOPOLOGY

Bernhart, Arthur. Six-rings in minimal five-color maps. Amer. J. Math. 69, 391–412 (1947).

A systematic notation is used to prove again the known theorems on 4- and 5-rings, and to study the structure of regions inside 6-rings. It is shown that if 6-rings exist in minimal maps, the structure must be of one of six types.

P. Franklin (Cambridge, Mass.).

Kelley, J. L. Simple links and fixed sets under continuous mappings. Amer. J. Math. 69, 348–356 (1947).

This paper was abstracted in Proc. Nat. Acad. Sci. U. S. A. 26, 192–194 (1940); these Rev. 1, 222.

D. W. Hall.

Gustin, William. Sets of finite planar order. Duke Math. J. 14, 51–66 (1947).

If E is a subset of Euclidean space of dimension n and every plane (any subspace of dimension $n-1$) intersects E in at most t points, then E is said to be of planar order t. If every plane intersection includes exactly t points, then E is of exact planar order t; if the set E is of planar order n it is said to be of minimal planar order. The following results are obtained. Every connected set of finite planar order is locally connected. The closure of a connected set of planar order t is of planar order $2t-2$. Every connected set of minimal planar order is a simple continuous curve in the

boundary of a convex set. A set of exact minimal planar order is totally disconnected and is not the sum of a countable number of closed sets. *V. W. Adkisson.*

Krasnosselsky, M. Sur un critère pour qu'un domaine soit étoilé. Rec. Math. [Mat. Sbornik] N.S. **19**(61), 309–310 (1946). (Russian. French summary)

If F is a closed and bounded subset of Euclidean n-space E_n, a point x of its boundary H is said to be accessible by a segment from a point y of F if the segment $[x, y]$ is entirely contained in F. It is proved that if every set of $n+1$ points of H are accessible by segments from some point of F then there is a point t of F such that every point of H is accessible by a segment from t. The proof, which is carried out for the plane, by a method capable of generalization to arbitrary E_n, makes uses of some elementary constructions and a theorem of E. Helly [Jber. Deutsch. Math. Verein. **32**, 175–176 (1923)]. *H. Wallman* (Cambridge, Mass.).

Elsgoltz, L. Sur la variation du groupe fondamental du domaine des valeurs inférieures d'une fonction définie sur une multiplicité. Rec. Math. [Mat. Sbornik] N.S. **19**(61), 237–238 (1946). (Russian. French summary)

Remarks are presented on the change in fundamental group of the domain of lower values ($f \leqq x$) of a function f defined on a manifold when a critical point is adjoined to ($f \leqq x$). In certain cases these remarks permit extension of M. Morse's results on the number of critical points; thus it can be said that on the Poincaré sphere a twice differentiable function without degenerate critical points must possess at least 6 critical points. *H. Wallman.*

Frenkel, Yanny. Criteria of bicompactness and of H-completeness in an accessible topological Fréchet-Riesz space. Unión Mat. Argentina. Memórias y Monografias (2) **2**, no. 1, 21 pp. (1946). (Spanish)

The paper also appeared in Ciencia y Técnica **107**, 383–401 (1946); these Rev. **8**, 285.

Dieudonné, Jean. Sur les groupes compacts d'homéomorphismes. Anais Acad. Brasil. Ci. **18**, 287–289 (1946).

A complete metric space is given together with a group H of homeomorphisms of E. If H has a topology satisfying certain conditions, then it is shown that this topology is identical with the topology which can be introduced in H by convergence on compact subsets. It is also shown that if H is a group of homeomorphisms of E satisfying conditions of equicontinuity then the closure of H is a compact topological transformation group of E. These results are closely related to results of Arens and of Myers [Ann. of Math. (2) **47**, 480–495, 496–502 (1946); these Rev. **8**, 165]. *D. Montgomery* (Princeton, N. J.).

Scorza Dragoni, G. Un teorema fondamentale sulle traslazioni piane generalizzate. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **1**, 1163–1166 (1946).

The author states without proof several results closely related to a theorem on fixedpoint-free automorphisms of the plane which he recently published [same Rend. (8) **1**, 697–704 (1946); these Rev. **8**, 285]. *P. A. Smith.*

Scorza Dragoni, G. Su alcune totalità di archi di traslazione di un autoomeomorfismo piano, conservante il senso delle rotazioni e privo di punti uniti. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **2**, 34–37 (1947).

A summary of the author's recently published results concerning fixedpoint-free automorphisms of the plane. [See Atti Accad. Italia. Mem. Cl. Sci. Fis. Mat. Nat. **9**, 1–75 (1937); same Rend. (8) **1**, 156–161 (1946); these Rev. **8**, 285.] *P. A. Smith* (New York, N. Y.).

***Tucker, A. W. Some topological properties of disk and sphere.** Proc. First Canadian Math. Congress, Montreal, 1945, pp. 285–309. University of Toronto Press, Toronto, 1946. $3.25.

An exposition of a chapter of topology in 2 and 3 dimensions dealing with fixed-point theorems, vector fields, covering theorems of different kinds and some applications. The development is mainly based on the well-known theorems on the sphere due to Borsuk [Fund. Math. **20**, 177–190 1933)], for which simple proofs are given. *W. Fenchel.*

Whitehead, J. H. C. An expression of Hopf's invariant as an integral. Proc. Nat. Acad. Sci. U. S. A. **33**, 117–123 (1947).

The main result of the paper is indicated by the title. Let $f: S^3 \to S^2$ be a twice differentiable map, where S^3, S^2 are spheres of dimensions three and two, respectively. To express the integral in question take the element of area of S^2, divided by 4π, and let ω_2 be the image of this differential form under the dual mapping f_* of f. Since the two-dimensional cohomology group of S^3 is trivial, there exists a linear differential form ω_1 such that its exterior derivative is ω_2. Then the Hopf invariant is equal to the integral of $\omega_1\omega_2$ over S^3. Complete proof of the formula is given. *S. Chern* (Shanghai).

Steenrod, N. E. Cohomology invariants of mappings. Proc. Nat. Acad. Sci. U. S. A. **33**, 124–128 (1947).

An operation is defined which greatly extends the Hopf invariant of a map of a $(2n-1)$-manifold on an n-manifold. Let K, K' be complexes, and $f: K \to K'$ a simplicial map. Then f induces homomorphisms f^* of the cohomology groups $H^p(K')$ (with integer coefficients) of K' into the cohomology groups $H^p(K)$ of K. Let $u \epsilon H^p(K')$, $v \epsilon H^q(K')$, such that $f^*u = 0$, $u \cup v = 0$. The operation of the author is to define, from the map f and the cohomology classes u, v, an element $[f, u, v] \epsilon H^{p+q-1}(K)/[f^*H^{p+q-1}(K') + H^{p-1}(K) \cup f^*v]$. If $f: S^3 \to S^2$ is a map of a 3-sphere into a 2-sphere and u is a generator of $H^2(S^2)$, then $[f, u, u] = \gamma z$, where z is a generator of $H^3(S^3)$ and γ is the Hopf invariant of f. Various properties of the operation are stated, among which are: (1) if f is homotopic to g, then $[f, u, v] = [g, u, v]$; (2) $[f, u, v]$ is linear in u and if $f^*v = 0$, then $[f, u, v] = (-1)^{pq}[f, v, u]$, etc. Several examples are given, showing that the operation serves to distinguish between maps of the same homology type. Extensions are stated, to relative cohomology groups and to an operation which involves the squaring operations $u \cup_i u$ of the author. Finally, an alternative way to define the operation is suggested, which makes the operation invariant by definition, so that proofs based on it are simpler. *S. Chern* (Shanghai).

GEOMETRY

Busemann, Herbert. Two-dimensional geometries with elementary areas. Bull. Amer. Math. Soc. 53, 402–407 (1947).

In this note the author makes use of the notations, definitions and results of his earlier memoirs [Metric methods in Finsler spaces and in the foundations of geometry, Ann. of Math. Studies, no. 8, Princeton University Press, 1942; Trans. Amer. Math. Soc. 56, 200–274 (1944); these Rev. 4, 109; 6, 97]. Let R denote a two-dimensional E-space. A symmetric function $\alpha(a_1a_2a_3)$ defined for triples a_1, a_2, a_3 is called an area if (A) $0\leqq\alpha(a_1a_2a_3)<\infty$ and $\alpha(a_1a_2a_3)=0$ if and only if a_1, a_2, a_3 are collinear; (B) if b lies between a_2 and a_3, then $\alpha(a_1a_2b)+\alpha(a_1ba_3)=\alpha(a_1a_2a_3)$. Theorem I. If (and only if) locally an area exists for which triangles with equal sides have equal area, then the space is a locally isometric map of either the Euclidean plane or a hyperbolic plane or a sphere. Theorem II. If (and only if) locally an area α exists such that the area of the triangle pab depends only on p, the local branch of the geodesic $\mathfrak{g}$ that contains the segment $\mathfrak{s}(a, b)$ and the distance ab, then the space is a locally isometric map of a Minkowski plane. *C. Pauc.*

Davids, Norman. A characterization of Minkowskian geometry. Bull. Amer. Math. Soc. 53, 196–200 (1947).

We denote by L a finitely-compact metric space in which any two distinct points are contained in exactly one set congruent to a Euclidean straight line. Let (OX), (OY) be two arbitrary segments in L having the endpoint O in common. We call the segment (OP) a sum of the two given segments, written $(OP)=(OX)+(OY)$, if the midpoint of (OP) coincides with the midpoint of (XY). The set of segments through O satisfies all the group axioms for addition except the associative law; the hyperbolic plane is an example where it fails to hold. It is shown in the present note that a space L where the associative law holds is a Minkowskian space. *C. Pauc* (Marseille).

Mendelsohn, N. S. A group theoretic characterization of the general projective collineation group. Trans. Roy. Soc. Canada. Sect. III. (3) 40, 37–58 (1946).

Die Gruppe Γ der linearen Transformationen dreier reeller homogener Veränderlichen enthält eine Klasse von Untergruppen P, gebildet aus den Transformationen, die alle Geraden durch einen gegebenen Punkt p festlassen, bzw. eine Klasse von Untergruppen L, gebildet aus den Transformationen, die alle Punkte auf einer gegebenen Geraden l festlassen. Die Untersuchung der Nomalisatoren, der Transformations- und Zerlegungseigenschaften dieser Untergruppen führt zur Aufstellung eines Systems von 6 Axiomen für die beiden Klassen von Untergruppen, aus dem sich eine ebene projektive abstrakte Geometrie gewinnen lässt: ein Punkt ist dabei eine Untergruppe P, eine Gerade eine Untergruppe L; Inzidenz von Punkt und Gerade ist dadurch definiert, dass P im Normalisator von L enthalten ist. In dieser abstrakten ebenen Geometrie lassen sich aus den zugrundegelegten Axiomen die projektiven Verknüpfungssätze beweisen. Die projektiven Abbildungen erscheinen als Transformationen der den Raumelementen zugeordneten Untergruppen in einander, und es gilt der Fundamentalsatz der projektiven Geometrie. Mit jedem Automorphismus der Collineationsgruppe Γ der abstrakten ebenen Geometrie ist ein-eindeutig eine Collineation bzw. Correlation der Ebene verknüpft. [Bereits M. Dehn hat in einer vom Verfasser nicht zitierten Arbeit [Mat. Tidsskr. B. 1931, 63–83] die nichteuklidischen Bewegungen gruppentheoretisch charakterisiert und darauf hingewiesen, dass die gruppentheoretische Definition kontinuierlicher Gruppen sich wesentlich von der gewöhnlichen Definition diskontinuierlicher Gruppen durch Erzeugende und Relationen als zweiseitige Verwandlungsregeln unterscheidet.] *R. Moufang.*

Gambier, Bertrand. Quelques réflexions à propos de la parataxie. Ann. Sci. École Norm. Sup. (3) 63, 23–44 (1946).

L'auteur reprend quelques points qu'il désire préciser, de son ouvrage antérieur [Cycles paratactiques, Memor. Sci. Math., no. 104, Gauthier-Villars, Paris, 1944; ces Rev. 7, 472]. Il reproduit quelques passages d'une lettre d'É. Cartan concernant la définition des foyers, la notion de droite isotrope, celle d'ordre anallagmatique et celle d'orientation. L'auteur étudie d'autre part le groupe à 20 paramètres des transformations qui changent un cycle réel en un cycle réel en conservant la parataxie, et une configuration de dix cycles réels tels que chacun d'eux est paratactique à quatre autres cycles de la configuration. *L. Gauthier.*

Sen Gupta, B. K. On the aberrancy curve. J. Indian Math. Soc. (N.S.) 10, 33–34 (1946).

Addendum to the author's paper in the same J. (N.S.) 9, 77–79 (1945); these Rev. 8, 344. In the original the word "the" was repeated in the title.

Bilo, J. Remarkable cubic curves in metrically special homaloidal nets. Simon Stevin 25, 69–82 (1947). (Dutch)

The author gives simple geometrical constructions for the birational transformations T, T' which transform the homogeneous coordinates x, y, z into axy^2, ay^3, $(x-az)(\pm x^2-y^2)$, first in the projective plane and then in the Euclidean plane (with Cartesian coordinates x/z, y/z). The lines of the plane are transformed by T^{-1}, T, T'^{-1}, T' into four "homaloidal nets" of cubic curves. The first is the net already described by Claeys [Wis- en Natuurk. Tijdschr. 12, 119–142 (1945); these Rev. 7, 320], which includes the folium of Descartes and the right strophoid. The second includes Rolle's curve $(x-y)^2(x+y)-ay^2z=0$, the third includes the cubic duplicatrix $x^3-a(x^2+y^2)z=0$ and the fourth includes the cissoids. *H. S. M. Coxeter* (Toronto, Ont.).

Thébault, Victor. Sur le cercle et la sphère de Hagge. Bull. Soc. Roy. Sci. Liége 15, 44–52 (1946).

Thébault, V. Sur les tétraèdres orthologiques. Ann. Soc. Sci. Bruxelles. Sér. I. 61, 112–116 (1947).

Thébault, V. Sur le cercle des orthopôles. Bull. Soc. Roy. Sci. Liége 14, 299–307 (1945).

Goormaghtigh, M.-R. Orthopôles et droites orthopolaires dans les polygones. Bull. Soc. Roy. Sci. Liége 15, 119–133 (1946).

Goormaghtigh, M.-R. Isopôles et droites isopolaires dans les polygones. Bull. Soc. Roy. Sci. Liége 15, 213–220 (1946).

Goormaghtigh, R. The Hervey point of the general n-line. Amer. Math. Monthly 54, 327–331 (1947).

Goormaghtigh, M. R. Sur un groupe de paraboles associées au triangle. Inst. Grand-Ducal Luxembourg. Sect. Sci. Nat. Phys. Math. Arch. N.S. 16, 96–97 (1946).

Algebraic Geometry

Kasner, Edward, and De Cicco, John. Curvature theorems on polar curves. Proc. Nat. Acad. Sci. U. S. A. 33, 47–51 (1947).

If P is a point on a plane algebraic curve, then the polars of P with respect to this curve also pass through P. For P ordinary, the ratios of the curvatures at P of these various polars are computed and found to be arithmetical invariants depending only on the order of the initial curve. The ratios of departure of these polar curves from the tangent line at a simple point P are also found to be arithmetical invariants. On the other hand, if P is a simple cusp having an integral order of contact with the tangent line, then the ratio of departure is no longer an arithmetical invariant, but still is a projective invariant. *A. Seidenberg* (Berkeley, Calif.).

Kasner, Edward, and De Cicco, John. The curvatures of the polar curves of a general algebraic curve. Amer. Math. Monthly 54, 263–268 (1947).

Let P be a simple point of a plane algebraic curve C_n of order n. Then the rth polar of P is a curve C_{n-r} of order $n-r$ which touches C_n at P. The authors prove that the ratio of the curvatures of C_{n-r} and C_n at P is $(n-r-1)/(n-1)$. *J. A. Todd* (Cambridge, England).

***Morin, Ugo. Sull'unirazionalità dell'ipersuperficie algebrica di qualunque ordine e dimensione sufficientemente alta.** Atti Secondo Congresso Un. Mat. Ital., Bologna, 1940, pp. 298–302. Edizioni Cremonense, Rome, 1942.

An irreducible algebraic variety of dimension d is unirational if it is not rational, but the coordinates of a generic point can be expressed as rational functions of d indeterminates. For $d\leqq 2$ unirationality implies rationality [Lüroth, $d=1$; Castelnuovo, $d=2$]. The following theorem is proved. The algebraic variety given by the single equation $f(x_0, \cdots, x_r)=0$, where f is a generic form of order n, is unirational if $r\geqq r_n$, where

$$r_n=\binom{n+r_{n-1}}{n},\quad r_2=2.$$

D. Pedoe (London).

Morin, Ugo. Sulla razionalità dell'ipersuperficie cubica generale dello spazio lineare S_5. Rend. Sem. Mat. Univ. Padova 11, 108–112 (1940).

An irreducible cubic hypersurface V_4^3 in an S_5 is rational if it contains a ruled rational normal F_2^4. This fact is used to prove that a general V_4^3 of S_5 is rational. The surfaces F_2^4 form an irreducible 29-dimensional algebraic system. [For the basic properties of these F_2^4 see E. Bertini, Einführung in die projective Geometrie mehrdimensionaler Raume, Vienna, 1924, chap. XIV.] The V_4^3 form a 55-dimensional linear system, which cut out on an F_2^4 a complete $|C^{12}|$ of dimension 27, so that a necessary and sufficient condition that a V_4^3 contain a given F_2^4 is that it contain 28 points of the F_2^4 in a general position. In the space S_{55} of the V_4^3, to the V_4^3 through a general F_2^4 there corresponds an S_{27}. The points of all these S_{27} form an irreducible $W_{55-\lambda}$, and it remains to prove that $\lambda\geqq 1$ is impossible. Using the method of counting constants one finds that through a general point of $W_{55-\lambda}$ there passes a system of the S_{27} of dimension $\lambda+1$, and that if two of the S_{27} meet in an S_ρ then $\rho\geqq\lambda-1$. Equality is excluded by the lemma that two general F_2^4 in nonspecial position cannot be contained in a V_4^3, while inequality is excluded by the fact that two F_2^4 having a point in common, but otherwise in nonspecial position, belong to one and only one V_4^3.

A. Seidenberg (Berkeley, Calif.).

Zappa, Guido. Sulla degenerazione delle superficie algebriche in sistemi di piani distinti, con applicazioni allo studio delle rigate. Atti Accad. Italia. Mem. Cl. Sci. Fis. Mat. Nat. 13, 989–1021 (1942).

In appendices F and G to his "Vorlesungen über algebraische Geometrie" [Leipzig, 1921] Severi has studied questions relative to the degeneration of a variable irreducible algebraic curve, of a given order n, into n lines. The present paper deals with similar questions in the case of algebraic surfaces. In the first part of the paper the author studies the degeneration of a general surface F in S_4 into n distinct planes. The crossings of the set L of these planes may be either lines or isolated points, and both are analyzed in relation to the connectivity features of the n-plane L regarded as a limit of F. The limit of the variety of the bisecants of F is derived. By projection into S_3 similar results are obtained for the degeneration of a surface Φ in S_3 (with ordinary singularities) into an n-plane Λ. The virtual geometric genus p_g and arithmetic genus p_a of Λ are calculated and the inequality $p_g\geqq p_a$ (for Λ, not necessarily for Φ) is derived. In the second part of the paper the author deals with the degeneration of a ruled surface into quadrics or into planes, under certain assumptions relative to the connectivity features of the degenerate surface. Known facts about ruled surfaces are verified on the degenerate limit surface, e.g., (1) $p_g=0$, $p_a=-p$; (2) $R_1=2p$; (3) the nonexistence of transcendental 2-cycles. *O. Zariski.*

Arvesen, Ole Peder. Note on s_n-curves and -surfaces and some of their applications. Norsk Mat. Tidsskr. 27, 1–9 (1945). (Norwegian)

Es sei (*) $\sum_{i=r}^{m} w^{m-i}\varphi_i(u, v)$ die Gleichung in homogenen Linienkoordinaten u, v, w einer ebenen algebraischen Kurve k der Klasse m, die die unendlich ferne Gerade r-fach berührt. Für jedes ganze n wird mit k die "s_n-Kurve" $(m-r)w^n=\sum_{i=1}^{m-r} w_i^n$ verknüpft, wo w_i die Wurzeln der Gleichung (*) sind. Bei passender Wahl der Koordinaten lässt sich diese Kurve so charakterisieren: Der Abstand des Nullpunktes von einer beliebigen Tangente t der s_n-Kurve ist das $n$te Potenzmittel der Abstände des Nullpunktes von den $m-r$ zu t parallelen eigentlichen Tangenten von k. Analog wird die s_n-Fläche einer algebraischen Fläche definiert. Der Verfasser diskutiert diese Begriffsbildung in einigen einfachen Fällen und verwendet sie zur Lösung von speziellen analytisch-geometrischen Aufgaben.

W. Fenchel (Kopenhagen).

Stubban, John Olav. Some properties of diameters of certain isotropic algebraic curves. Norsk Mat. Tidsskr. 25, 33–36 (1943). (Norwegian)

Der Verfasser betrachtet ebene isotrope algebraische Kurven (d.h. Kurven, die nur die Kreispunkte mit der unendlich fernen Geraden gemein haben), deren Durchmesser erster Ordnung alle durch einen festen Punkt, den sogenannten Fundamentalpunkt, gehen. Es werden einige Eigenschaften der Asymptoten der Durchmesser beliebiger

Ordnung abgeleitet, z.B. dass diese Asymptoten durch den Fundamentalpunkt gehen. *W. Fenchel* (Kopenhagen).

Stubban, John Olav. Généralisation d'une formule de Reiss. Norske Vid. Selsk. Forh., Trondhjem **16**, no. 21, 76–79 (1943).

La formule de Reiss [voir Pascal, Repertorium der höheren Geometrie, 2ᵉ ed., t. 2, Leipzig-Berlin, 1910, p. 431] relative à l'intersection d'une courbe algébrique C et d'une transversale D, relie les angles θ et les rayons de courbure R aux points communs: $\sum 1/(R\sin^3\theta)=0$. L'auteur étend cette formule à l'intersection d'une courbe algébrique C et d'une courbe de direction algébrique Δ:

$$\sum\left(\frac{\cos\theta}{r}-\frac{1}{R}\right)\frac{1}{\sin^3\theta}=0,$$

où r est le rayon de courbure de Δ, R celui de C et la sommation étendue à tous les points communs. *L. Gauthier.*

Stubban, John Olav. Sur les transformations birationelles dans la géométrie de direction. Norske Vid. Selsk. Forh., Trondhjem **17** (1944), no. 36, 142–145 (1945).

Les demi-droites du plan peuvent être représentées par les points d'un espace à trois dimensions Cayleyen Σ dont l'absolu est un cône réel du second ordre C. L'auteur étudie les transformations $\mathcal{T}$ de la géométrie de direction représentées par les transformations birationnelles T de Σ qui conservent l'absolu C; si $\mathcal{T}$ transforme une courbe de direction de classe n en une courbe de classe $n'=kn$ on dit que $\mathcal{T}$ est d'ordre k. Bien que les transformations inverses T et T^{-1} aient des ordres $\mu\mu'$ en général différents, les transformations $\mathcal{T}$ et $\mathcal{T}^{-1}$ ont le même ordre k, et on a $k=\omega+1$, ω étant la multiplicité du sommet du cône C pour les surfaces transformées des plans. *L. Gauthier* (Nancy).

Amodeo, Federico. Nuovo metodo per la geometria delle serie lineari delle curve algebriche. Rend. Accad. Sci. Fis. Mat. Napoli (4) **9**, 21–44 (1939).

Longhi, A. Sulle involuzioni cubiche di 2ᵃ specie. Elemente der Math. **2**, 28–30 (1947).

Seifert, L. L'hypersurface cubique à un point double conique dans l'espace à quatre dimensions et l'ensemble de surfaces cubiques passant par une courbe donnée du degré six et du genre quatre. Acad. Tchèque Sci. Bull. Int. Cl. Sci. Math. Nat. **40**, 215–217 (1939).

Godeaux, Lucien. Sur la construction de surfaces algébriques contenant une involution cyclique. Bull. Soc. Roy. Sci. Liége **14**, 64–69 (1945).

Godeaux, Lucien. Sur la construction de surfaces algébriques triples. I. Bull. Soc. Roy. Sci. Liége **14**, 259–268 (1945).
Godeaux, Lucien. Sur la construction de surfaces algébriques triples. II. Bull. Soc. Roy. Sci. Liége **14**, 274–282 (1945).

Godeaux, Lucien. Sur quelques variétés réglées à trois dimensions. Bull. Soc. Roy. Sci. Liége **14**, 470–476 (1945).

Godeaux, Lucien. Observations sur les surfaces algébriques de rang trois. Bull. Soc. Roy. Sci. Liége **15**, 2–8 (1946).

Godeaux, Lucien. Quelques remarques sur les surfaces de genres un et de rang deux. I. Bull. Soc. Roy. Sci. Liége **14**, 282–296 (1945).
Godeaux, Lucien. Quelques remarques sur les surfaces de genres un et de rang deux. II. Bull. Soc. Roy. Sci. Liége **14**, 332–341 (1945).
Godeaux, Lucien. Quelques remarques sur les surfaces de genres un et de rang deux. III. Bull. Soc. Roy. Sci. Liége **14**, 398–402 (1945).

Godeaux, Lucien. Sur les surfaces algébriques ayant une courbe double tracée sur une surface du quatrième ordre. Bull. Soc. Roy. Sci. Liége **15**, 54–62 (1946).

Godeaux, Lucien. Sur certaines involutions cycliques du quatrième ordre appartenant à une surface de genres un. Bull. Soc. Roy. Sci. Liége **15**, 163–166 (1946).

Godeaux, Lucien. Sur une involution rationnelle du septième ordre appartenant à une surface irrégulière. Bull. Soc. Roy. Sci. Liége **15**, 278–284 (1946).

Jongmans, F. Un théorème sur les polaires des surfaces de l'espace à trois dimensions. Bull. Soc. Roy. Sci. Liége **14**, 431–436 (1945).

Lorent, H. Sur des surfaces associées à une biquadratique gauche de première espèce. Bull. Soc. Roy. Sci. Liége **14**, 297–299 (1945).
Lorent, H. Sur des surfaces associées à une biquadratique gauche de première espèce. II. Bull. Soc. Roy. Sci. Liége **14**, 341–346 (1945).
Lorent, H. Sur des surfaces associées à une biquadratique gauche de première espèce. III. Bull. Soc. Roy. Sci. Liége **14**, 411–419 (1945).

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d'Orgeval, B. Une limite supérieure du nombre de certaines surfaces rationnelles. Bull. Soc. Roy. Sci. Liége **15**, 41–44 (1946).

d'Orgeval, Bernard. Remarques sur des plans quadruples dont les courbes de diramation possèdent les mêmes caractères, mais des décompositions de Chisini distinctes. Bull. Soc. Roy. Sci. Liége **15**, 205–207 (1946).

Pétermans, F. Sur une surface normale du septième ordre à sections de genre quatre. Bull. Soc. Roy. Sci. Liége **14**, 182–186 (1945).

Deprez, Henri. Transformation birationnelle associée à une surface d'ordre n ayant un point multiple d'ordre $n-2$. Bull. Soc. Roy. Sci. Liége **14**, 176–181 (1945).

Derwidué, L. Sur les courbes unies multiples des transformations birationnelles planes. Bull. Soc. Roy. Sci. Liége **14**, 354–365 (1945).

Derwidué, L. Sur une nouvelle transformation admettant une sextique rationnelle unie. Bull. Soc. Roy. Sci. Liége **14**, 425–430 (1945).

Derwidué, L. Sur une transformation plane admettant une courbe unie d'ordre neuf, douée de dix points triples. Bull. Soc. Roy. Sci. Liége **15**, 31–36 (1946).

Derwidué, L. Transformations birationnelles planes dont la courbe unie est une sextique (dégénérée) douée de onze points doubles. Bull. Soc. Roy. Sci. Liége **15**, 36–41 (1946).

Derwidué, L. Sur les courbes et les surfaces unies multiples des transformations birationnelles de l'espace. Bull. Soc. Roy. Sci. Liége **15**, 68–76 (1946).

Derwidué, L. Sur une classification des transformations birationnelles de l'espace. Bull. Soc. Roy. Sci. Liége **15**, 208–213 (1946).

Wiser, Pierre. Sur une transformation birationnelle de l'espace. Bull. Soc. Roy. Sci. Liége **15**, 382–389 (1946).

Differential Geometry

Câmpan, Florica. Surfaces parallèles et semblables. Disquisit. Math. Phys. **3**, 85–117 (1943).

In his investigation of curves which are both parallel and similar Teixeira [Intermédiaire des Math. **19**, 154–156 (1912)] showed that all the curves parallel to a logarithmic spiral are mutually similar, but that they are not similar to the spiral itself. Furthermore, any two parallel and similar curves are the involutes of a logarithmic spiral. Finally this class of curves is the most general set of parallel and similar curves; hence this subject in the plane is entirely settled.

This paper extends the above results to surfaces which are parallel and similar. The spiral is replaced by the "spiral surface" $\rho=e^{k\omega}f(\theta)$, where ω is the longitude and θ the colatitude. With this change the plane theory goes over to surfaces almost word for word. First the author shows that the involutes of the spiral surface generate all parallel and similar surfaces and hence derives parametric equations of these surfaces. These involve a function φ which is determined only as a solution of a differential equation. The second focal nappe of this family is also a spiral surface. Another set of parametric equations, this time completely determined, are also developed by considering the family of surfaces parallel to the spiral surface. These two sets of equations are shown to be equivalent.

The last portion of the paper deals with the special case in which the evolute spiral surface is a ruled surface. In this case the function φ can be determined by integration and specific equations for the desired surfaces are obtained. A collateral result is the determination of the nature of the traces of a spiral ruled surface in the meridian planes $\omega=$constant. *C. B. Allendoerfer* (Haverford, Pa.).

Rollero, A. Punto flecnodale delle superficie. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **2**, 30–33 (1947).

The object of this paper is to derive a canonical power series expansion of a surface S at a flecnodal point. The expansion found has the form $Z=xy-y^3+I(x-4y)x^3+[5]$, I being a unique invariant. The geometrical characterization of the canonical tetrahedron giving rise to this expansion is made to depend on the asymptotic tangents, the pencil of quadrics having second order contact with S at the flecnode and a covariant cubic representing the third order neighborhood of S at that point. *V. G. Grove.*

Barba, Guido. Sulla configurazione dei cerchi osculatori e delle sfere osculatrici. Rend. Accad. Sci. Fis. Mat. Napoli (4) **11**, 81–92 (1941).

It is shown that the osculating circles of a curve in the plane at two infinitesimally near points do not have any real points in common; moreover, one circle is in the interior of the other. This is valid along an analytic arc where the radius of curvature does not assume its maximum or minimum values. The corresponding situation for a twisted space curve is studied. The osculating circles at two infinitesimally adjacent points have no real points in common. However, adjacent osculating spheres do intersect. Of course, the limiting position of the circle of intersection is the osculating circle. *J. De Cicco* (Chicago, Ill.).

Silva, Giovanni. Contributo allo studio di alcuni enti geometrici nei punti di una linea sghemba. Ist. Veneto Sci. Lett. Arti. Parte II. **104**, 1053–1080 (1946).

The author studies the differential geometry of an analytic curve of ordinary Euclidean space in the neighborhood of a singular point. Let the vector equation of a curve C be $P=P(t)=(x(t), y(t), z(t))$, where t is a parameter. Consider a point $P_0=P(t_0)$ on C where the first $i-1$ derivatives vanish but $P_0^{(i)}\neq 0$; the $k-i-1$ vectors $P_0^{(i+1)}, \cdots, P_0^{(k-1)}$ are zero or collinear with $P_0^{(i)}$, but $P_0^{(k)}\neq 0$, is not collinear with $P_0^{(i)}$; the $(l-k-1)$ vectors $P_0^{(k+1)}, \cdots, P_0^{(l-1)}$ are zero or coplanar with $P_0^{(i)}$ and $P_0^{(k)}$, but $P_0^{(l)}\neq 0$, does not belong to the plane determined by $P_0^{(i)}$ and $P_0^{(k)}$. It is found that the following inequalities are not valid for all values of t in a neighborhood of t_0: (1) $i>1$ unless P_0 is an isolated point, (2) $k>i+1$ unless P describes a straight line segment, (3) $l>i+2$ unless P describes a plane arc.

Under the above hypotheses, it is found that the unit tangent vector at P_0 is $t_0=P_0^{(i)}/|P_0^{(i)}|$, the unit binormal vector is $b_0=P_0^{(i)}\times P_0^{(j)}/|P_0^{(i)}\times P_0^{(j)}|$ and the unit normal vector is $n_0=b_0\times t_0$. The equation of the osculating plane is $(P-P_0, P_0^{(i)}, P_0^{(k)})=0$. Formulas are given for the radius of curvature ρ_0 and the radius of torsion τ_0 at P_0. Upon interpreting t as the time so that the curve C is the trajectory of a particle P, the author also develops formulas for the vector velocity and acceleration at a singular point P_0. *J. De Cicco* (Chicago, Ill.).

Silva, Giovanni. Concavità, convessità e curvatura di una linea relative a un punto dato e a una data direzione. Ist. Veneto Sci. Lett. Arti. Parte II. **104**, 1081–1096 (1946).

At a point P of a curve l, construct the center of curvature C. The curve l is concave or convex at P with respect to a fixed point O according as the angle CPO is acute or obtuse. By letting O recede to infinity along a fixed direction m, the analogous definition with respect to the direction m is obtained. Let $P=P(t)$ be the vector equation of the curve l, where t is the independent parameter defining points P on l. Let M be a unit vector with the direction PO, where O is a fixed point or PO has a fixed direction m. The curve l is concave or convex with respect to O or m according as the scalar $(P'\times P'')\times P'\cdot M$ is positive or negative.

Consider the cone Γ with vertex at O and generators OP, where P is a point of l. Let r be the distance OP and θ the angle on the cone Γ between the variable generator OP and a fixed generator OP_0 on the cone, let s be the arc length of l from P_0 to P and let ϕ be the angle between the radius vector OP and the tangent line to l at P. The formulas

$\tan\phi = rd\theta/dr$, $\sin\phi = rd\theta/ds$, $\cos\phi = dr/ds$, analogous to the corresponding ones for a plane curve in polar coordinates, are derived. The curve l is concave or convex with respect to O according as the two expressions $1-\frac{1}{2}(d^2(r^2)/ds^2)$ and $1+r(d^2(1/r)/d\theta^2)$ are positive or negative. If both expressions are zero, the point O is in the rectifying plane of l at P or else P is a singular point.

The preceding formulas may be given geometrical interpretations. Let K be defined by $|P'|^4K=(P'\times P'')\times P'\cdot M$, where the primes denote differentiation with respect to the parameter t. The author defines K as the relative curvature of l at P with respect to O or m. It is found that K is the component of the vector curvature kn (k is the circular curvature and n is the unit principal normal vector) along the vector M. *J. De Cicco* (Chicago, Ill.).

✱Facciotti, Guido. Concavità o convessità in un punto di una curva sghemba rispetto ad un piano o rispetto ad un punto. Atti Secondo Congresso Un. Mat. Ital., Bologna, 1940, pp. 254–258. Edizioni Cremonense, Rome, 1942.

Gheorghiev, Gh. Sur certaines courbes dérivées d'une courbe gauche et applications. Bull. École Polytech. Jassy [Bul. Politehn. Gh. Asachi. Iaşi] 1, 225–233 (1946).

The formulae of curvature and torsion are developed for a curve on a developable surface consisting of the tangents of a given curve. These formulae are applied to certain satellite curves of given curves and to a generalization of the helicoidal surface. *J. A. Schouten* (Epe).

Lalan, Victor. Sur le système de Pfaff Codazzi-Gauss. C. R. Acad. Sci. Paris 224, 1201–1203 (1947).

The Pfaffian system equivalent to the equations of Gauss and Codazzi recently introduced by the author [same C. R. 224, 518–520 (1947); these Rev. 8, 404] can be written in various forms by means of certain supplementary hypotheses. In this paper the author considers three of these, namely: (1) where the surface is not of constant mean curvature; (2) where condition (1) is satisfied and the surface is not a-surface of Weingarten; (3) where (1) and (2) are satisfied and with suitable parameters $\partial L/\partial v=\partial N/\partial u$. The results in all cases are simplified differential equations which may be useful in further investigations.
C. B. Allendoerfer (Haverford, Pa.).

Bompiani, E. Piccoli contributi alla teoria gaussiana delle superficie. Boll. Un. Mat. Ital. (3) 1, 35–39 (1946).

The author obtains a geometric interpretation of the two Mainardi-Codazzi relations. Let the lines of curvature be chosen as parametric curves u (v=constant) and v (u=constant) and let R_1 and R_2 be the corresponding radii of curvature, so that $K=1/R_1R_2$. Consider a quadrilateral bounded by the lines of curvature with vertices $O(u, v)$, $O_u(u+du, v)$, $O_v(u, v+dv)$, $O_{uv}(u+du, v+dv)$. One of the Mainardi-Codazzi relations is equivalent to the fact that the Gaussian curvature K is equal to the product of the principal curvature $1/R_1$ calculated at O by the quotient of the difference in the angle between the normal lines at O_v and O_{uv} and the angle between the normal lines at O and O_v by the difference in the lengths of the sides O_vO_{uv} and OO_u of the quadrilateral. The other relation is interpreted geometrically in a similar fashion. Also two quantities are obtained which are invariant under isometric deformations which preserve the lines of curvature.

In the final part of the paper the author considers an extension of the Beltrami-Enneper theorem, which states that the square of the torsion of an asymptotic line is the negative of the Gaussian curvature K, to curves which are tangent to the asymptotic lines. If any such curve has second order contact with its asymptotic tangent, its torsion is given by the same formula. Finally there is derived a quantity invariant under those isometric deformations for which asymptotic tangents are unaltered. *J. De Cicco.*

Bompiani, E. Sulle varietà a k dimensioni contenenti almeno ∞^k rette. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 1, 1001–1005 (1946).

Togliatti has classified the V_4 with ∞^4 lines into three types, one of which had been previously studied by Scorza. The author has obtained more direct methods which not only solve this problem more readily, yielding additional properties, but are applicable to the solution of analogous problems. *T. R. Hollcroft* (Aurora, N. Y.).

Fong, Shu-Chu. On principal lines and principal points. Bull. Amer. Math. Soc. 53, 408–416 (1947).

Étant donné deux courbes gauches C et $\bar{C}$ qui se coupent ou qui sont tangentes en un point ordinaire, l'auteur appelle correspondance G celle qui associe sur ces courbes deux points P et $\bar{P}$ respectivement où les tangentes sont concourantes. Il est clair que la droite $P\bar{P}$ engendre, lorsque P décrit C, une développable. C'est essentiellement l'étude de ces correspondances G qui est traitée dans cet article. L'auteur retrouve, par ce moyen, les résultats développés par Palozzi [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) 7, 321–325 (1928)] relatifs à l'existence de points principaux et de droites principales. De plus, dans le cas où les deux courbes sont tangentes et où le plan principal (ou plan de Halphen) et les plans osculateurs des deux courbes se confondent, cas où points et droites principaux cessent d'exister, l'auteur fait apparaître un nouveau point covariant. *M. Decuyper* (Lille).

Rozet, O. Sur le réseau formé par les lignes de courbure d'une surface. Bull. Soc. Roy. Sci. Liége 15, 294–298 (1946).

Rozet, O., et Bonnet, F. Sur la théorie des surfaces et les suites de Laplace. Bull. Soc. Roy. Sci. Liége 14, 403–410 (1945).

A study is made of the sequence of Laplace arising from the lines of curvature of a nondevelopable surface. Conditions are found for the termination of the sequence at any stage. *V. G. Grove* (East Lansing, Mich.).

Hsiung, Chuan-Chih. A study on the theory of conjugate nets. Amer. J. Math. 69, 379–390 (1947).

This paper is concerned with the study of the projective differential geometry of curves of a conjugate net on an analytic nonruled surface in ordinary space. The surface S is referred to the conjugate net as parametric. A correspondence between a line l_1 which passes through the point P_x of S but does not lie in the tangent plane to S at P_x and a line l_2 which lies in the tangent plane of S at P_x is defined as follows. Let C_λ be a curve of S through P_x. As P_x moves along C_λ the u and v-parametric tangents generate ruled surfaces $R^{(u)}$, $R^{(v)}$. The plane determined by l_1, and the u (v)-tangent touches the ruled surface $R^{(u)}$ ($R^{(v)}$) at a unique point T ($\bar{T}$) of the u (v)-tangent. The line l_2 joining the points T, $\bar{T}$ is thus associated with the line l_1. [This correspondence is the analogue for a conjugate parametric net of the R_λ associate of the reciprocal of l_1 introduced by the re-

viewer, Trans. Amer. Math. Soc. **46**, 387–409 (1939), in particular, pp. 390–391; these Rev. **1**, 85.] The correspondence at each point P_x of C_λ is a polarity with respect to a quadric if and only if C_λ is a curve of the associate conjugate net of the parametric net. Two quadrics are introduced of which one has contact of the second order with $R^{(u)}$ at T and contact of the first order with $R^{(v)}$ at $\bar{T}$, and the other has these properties with the roles of u, v interchanged.

A new correspondence $\mathfrak{C}_k$ between lines l_1 and l_2 is defined as follows. Let g denote the polar line of any line l_1 with respect to any quadric of Darboux. This correspondence is $\mathfrak{C}_\infty$ and $P_1^{(\infty)}$, $P_2^{(\infty)}$ denote the intersections of g with the u and v-tangents to S at P_x, respectively. Let Γ_1 (Γ_2) denote the intersection of S by the plane determined by the line l_1 and the u (v)-tangent of S at P_x and let P_1^0 (P_2^0) denote the pole of l_1 with respect to the 4 point conics of Γ_1 (Γ_2) at P_x. Now select points P_1^k, P_2^k on the u, v-tangents, respectively, defined by the cross-ratio equations $(P_m^\infty P_m^0 P_x P_m^k)=k$, $m=1, 2$. A correspondence $\mathfrak{C}_k$ is thus defined, between a line l_1 and the line l_2 joining P_1^k, P_2^k, which is a polarity with respect to any quadric Q_k of a pencil $Q_k(k_4)$. The quadric for which $k=4/3$, $k_4=0$ is the canonical quadric of Davis. The quadrics $Q_\infty(k_4)$, $Q_0(k_4)$ are, respectively, the quadrics of Darboux and the quadrics having contact of third order with the parametric curves of S at P_x. The polar lines of the axis and the ray of the parametric conjugate net with respect to the quadric Q_k are, respectively, the canonical lines of the second and first kinds of Davis. By a proper selection of the constant k any desired line of either kind may be characterized. Finally the author makes use of the polarity of a line l_1, with respect to the quadrics Q_k and of the four point conics of projected u and v-curves (from a point of l_1 on the osculating planes of the u and v-curves, respectively) to define two one-parameter families of lines which he calls the first and second families. The definitions of the two lines of these families for $k=4$ are analogous to those of the canonical edges of Green, the distinction being that these lines are determined by a given conjugate net of S whereas the canonical edges of Green are determined by the asymptotic net of S. *P. O. Bell* (Lawrence, Kan.).

Kosmina, T. Transformation de Laplace des systèmes de surfaces triplement conjuguées. C. R. (Doklady) Acad. Sci. URSS (N.S.) **55**, 183–185 (1947).

A study of the Laplace transforms of a system C_3 of triply conjugate surfaces is made by Cartan's method of projective displacements. Three of the edges of the local tetrahedron of reference are the tangents to the conjugate nets of the system C_3. Typical theorems are as follows. Application of the transformation of Laplace to the surfaces of a system C_3 gives rise to a new system C_3. One may associate to each Laplace sequence of C_3 relative to one of its conjugate nets a family of inscribed sequences C_3 for each of which the third edge cuts two consecutive edges of the same kind of the original sequence. This family depends on two arbitrary functions of two parameters. *V. G. Grove.*

Charrueau, André. Sur des congruences de droites déduites d'une même surface et sur une transformation de contact qui se rattache à ces congruences. Bull. Sci. Math. (2) **70**, 127–148 (1946).

Consider a nondevelopable surface S and a unit vector I with initial point at O. Project any point A of S onto the plane σ through O perpendicular to I. Rotate this projection in a positive sense through 90° obtaining a point M. Through M draw a line d parallel to the normal to S at A. As A varies over S, the line d generates a congruence C; it is said that C is deduced from S by the construction of Guichard relative to I. The congruence C has σ as its mean surface and its developables correspond to the asymptotic curves on S. Let a second congruence C' be constructed in the same manner from a vector I' with the same initial point O; C and C' are congruences of Ribaucour and hence enjoy the properties of such congruences.

Other properties of C and C' proved in this paper may be stated as follows. Let B be the point of intersection of the lines joining the corresponding focal points of d and d'; the point B is that point in which the plane π determined by d, d' touches its envelope Σ. The plane π is parallel to the normal to S at A and hence the normals to S at A and to Σ at B are perpendicular. The plane through the line OB perpendicular to the plane π (of I and I') is parallel to the normal to S at A; the normals to S at A and to Σ at B are perpendicular.

The transformation of S into Σ by the process described is a contact transformation T of the second class. A detailed study of T is made; a typical theorem may be stated as follows. Let a trihedron be constructed having two edges coinciding with the bisectors of the angles between d and d'; revolve this tetrahedron around one of these two bisectors (considered fixed). One of the congruences C (or C') remains fixed. A one-parameter family Φ of surfaces Σ is obtained from S by applying T for each position of the trihedron. As A varies over S, the point B describes a line δ and the plane tangent to all of the surfaces Σ of Φ at the point B common to that surface and δ always passes through the line d (used in the generation of the congruence C). Moreover, as A varies over S, the line δ generates a congruence Γ whose developables also correspond to the asymptotic curves on S; the developables of Γ cut each surface of the family Φ in a conjugate net, one of the focal planes of the line δ (generating Γ) passes through one of the focal points on d, the other through the other focal point on d.

V. G. Grove (East Lansing, Mich.).

Bouligand, Georges. Théorie des surfaces et topologie restreinte du second ordre. C. R. Acad. Sci. Paris **224**, 1261–1263 (1947).

The author introduces the differential geometry of the group G_2 of twice differentiable homeomorphisms of 3-space. The tangent plane to a surface S at a point p (in the sense of ordinary differential geometry) is replaced by a surface which has contact of order one with S at p; similarly osculating parabolas and spheres are replaced by quasi-quadratic surfaces, in a manner invariant under G_2. Analogues for the classical concepts of conjugate and orthogonal directions, of asymptotic lines and lines of curvature are defined and analogues of classical theorems stated. *H. Samelson.*

Haimovici, Adolf. Sur la géométrie d'un groupe de contact. Ann. Sci. Univ. Jassy. Partie I. **29** (1943), 101–139 (1946).

The author studies the geometry of the four-parameter group of contact transformations of lineal elements in the plane consisting of Euclidean motions and dilatations. A dilatation is a contact transformation whereby each lineal element is displaced through a constant direction parallel to itself. This is a subgroup of the six-parameter group, the whirl-motion group G_6, consisting of motions and whirls, which has been studied extensively by Kasner and De Cicco. [See Kasner, Amer. J. Math. **33**, 193–202 (1911); De Cicco, Trans. Amer. Math. Soc. **46**, 348–361 (1939); **47**, 207–229

(1940); these Rev. 1, 84, 170.] The differential invariants of a series of lineal elements under this four-parameter group are found. It is shown that the only differential invariant of a curve is $dR/d\phi$, where R is the radius of curvature and ϕ is the angle that the tangent to the curve makes with a fixed direction.

The author also develops the geometry of the seven-parameter group of contact transformations of surface elements in space consisting of Euclidean motions and planar dilatations. Another proof is given of the theorem that the ratio of the curvature and torsion of the edge of regression of a developable surface is invariant under dilatation. Finally the author obtains the differential invariants of double series of elements consisting of ∞^2 surface elements under this seven-parameter group. *J. De Cicco.*

De Cicco, John. Union-preserving transformations of higher order surface-elements. Amer. J. Math. 69, 104–116 (1947).

Union-preserving transformations in 3-space from curve-elements of order n into lineal-elements have been studied by Kasner and the author [Bull. Amer. Math. Soc. 50, 98–107 (1944); Proc. Nat. Acad. Sci. U. S. A. 32, 152–156 (1946); these Rev. 5, 155; 7, 528]. The concept of a union-preserving transformation is now extended to include correspondences between surface-elements, particularly between surface-elements of order n and planar-elements. The surface-elements are x, y, z, p_{rs}, where $p_{rs}=\partial^{r+s}z/\partial^r x\partial^s y$ ($r, s=0, 1, \cdots, n$; $r+s\leqq n$) and the planar-elements are X, Y, Z, P, Q, where $P=\partial Z/\partial X$ and $Q=\partial Z/\partial Y$. It is shown that there are three kinds of nondegenerate union-preserving transformation, general, intermediate and special; a transformation is intermediate if it carries every conical union into a strip, special if it carries every conical union into a conical union and otherwise general. The three cases general, intermediate and special are characterized by the existence of respectively one, two and three directrix equations of the form $\Omega(X, Y, Z, x, y, z; \cdots, p_{rs}, \cdots)=0$, $r+s\leqq n-1$, a result which extends a theorem of Lie on contact transformations.

Finally, after considering transformations between surface-elements of higher orders, it is proved that the only union-preserving transformations from surface-elements of order $n\geqq 1$ into surface-elements of order m, where $n\geqq m\geqq 1$, are first, the contact group of planar-elements of Lie and second, the union-preserving transformations from surface-elements of order n, where n is 2 or more, into planar-elements, together with the extensions of these two types. *A. G. Walker* (Sheffield).

Haimovici, Mendel. La géométrie des familles de transformations de variables dépendant de paramètres. C. R. Acad. Sci. Paris 223, 969–971 (1946).

The author refers to former papers where he has proved that to every simply transitive set (famille) of transformations in n variables x^h, $h=1, \cdots, n$, an affine connection in a $(2n)$-dimensional space can be made to correspond (in several ways). Now he generalizes this result for multiply transitive sets. The set is supposed to be given by n Pfaff equations in the x^h and N parameters a^α, $\alpha=1, \cdots, N\geqq n$. Then an affine connection can be established uniquely by means of three geometric axioms and one axiom concerning the curvature affinor and the affinor of asymmetry, provided that a tensor of valence two, derived from this latter affinor, has rank $N-n$. A necessary condition is that $N\leqq n^2+n$. Curvature zero characterizes the groups. In this case the connection is identical with the first affine connection of Cartan. *J. A. Schouten* (Epe).

Woinaroski, Rudolf. Contributions à la cinématique des systèmes rigides d'un espace euclidien à quatre dimensions. Disquisit. Math. Phys. 4, 175–239 (1945).

Le but de ce travail est de trouver des éléments géométriques qui puissent remplacer les éléments cinématiques du mouvement d'un système rigide dans l'espace euclidien à quatre dimensions. En projetant sur un hyperplan V_3 les vitesses de ces points, on obtient un champ w de vitesses qui correspond au mouvement rigide de V_3 dans lui-même. L'ensemble des V_3 avec les axes, représentant les axes des mouvements hélicoïdaux instantanés, déterminés par les champs w, constitue un système focale. Si les variétés V_3 forment un faisceau ayant une V_2 commune laquelle contient le centre instantané de rotation, leurs axes forment une variété linéaire U_2. La relation entre V_2 et U_2 est réciproque. L'auteur donne plusieurs propriétés géométriques du système focale. Il montre comment on peut former d'une manière géométrique un système focale associé à un mouvement rigide.

La deuxième partie du travail contient la recherche d'éléments géométriques moyennant lesquels la reconstitution du mouvement rigide est possible. En registrant les positions successives du centre instantané du mouvement d'une part dans l'espace fixe et d'autre part dans le système rigide en mouvement on obtient deux courbes C et Γ. Ces courbes ne suffisent pas pour refaire le mouvement. Deux surfaces réglées, l'une dans l'espace mobile et l'autre dans l'espace fixe, sont définies et l'auteur montre comment ces deux surfaces avec les deux courbes C et Γ, situées sur ces surfaces, permettent la reconstitution du mouvement. On trouve les expressions des composantes du tenseur moyennant lequel on exprime les composantes des vitesses en fonction des éléments géométriques qui déterminent les deux surfaces. Avec cette méthode l'auteur traite le même problème pour le cas d'un surface à trois dimensions. *J. Haantjes* (Amsterdam).

Vyčichlo, F. Über die äquipollente Übertragung in der Geometrie von Möbius in der Ebene. Acad. Tchèque Sci. Bull. Int. Cl. Sci. Math. Nat. 44, 367–376 (1943).

With the aid of tetracyclical coordinates the circles with orientation (points and straight lines included) in a plane can be mapped on the points of a hypersurface in an R_4 with signature $+++-$. A curve $x^i=x^i(s)$ on this hypersphere represents a system of ∞^1 circles. The x^i, $i=1, \cdots, 4$, satisfying the condition $a_{ij}x^ix^j=1$, can be used as supernumerary coordinates on the sphere. To every linear element of the curves belongs an infinitesimal rotation in R_4. Now another system y^i of ∞^1 circles is considered from which each circle corresponds to a circle of the first system: $y^i=y^i(s)$. Then we may say that this second system is pseudoparallel in itself with respect to the first system if in R_4, at every point of the curve $x^i=x^i(s)$, y^i+dy^i is obtained from y^i by the same rotation as x^i+dx^i from x^i. A necessary and sufficient condition is

$$\frac{Dy^i}{ds}\equiv\frac{dy^i}{ds}+x^ia_{kl}dx^ky_l-\frac{dx^i}{ds}a_{kl}x^ky^l=0.$$

The geometrical significance in circle-language is given. The author calls Dy^i/ds the absolute conformal derivative of the system $y^i(s)$. The question of reciprocity between $x^i(s)$ and $y^i(s)$ is not considered but a continuation is promised. *J. A. Schouten* (Epe).

Mikan, Milan. Die Räume von R. König die den eingebetteten Mannigfaltigkeiten zugeordnet sind. Acad. Tchèque Sci. Bull. Int. Cl. Sci. Math. Nat. **40**, 218–221 (1939).

In an X_n an X_n^m and an X_n^p, $p=n-m$, are given. If the König space K_N, $N=n^r-1$, is built from the X_n [V. Hlavatý, Rend. Circ. Mat. Palermo **59**, 1–39 (1935)] the anholonomic X_n^m and X_n^p give rise to subspaces of K_N. The relations between the linear connexions in these different spaces are investigated. The paper seems to be an abstract of a longer paper. *J. A. Schouten* (Epe).

Wagner, V. Geometry of the n-dimensional space with the m-dimensional metric and its applications to the calculus of variations. Rec. Math. [Mat. Sbornik] N.S. 20(62), 3–26 (1947). (Russian. English summary)

This is a continuation of the author's work on the geometry of spaces with areal metric [same Rec. N.S. **19**(61), 341–406 (1946); these Rev. **8**, 490]. The space of all oriented m-directions in E_n constitutes a space $X_{m(n-m)}$. An m-dimensional Euclidean metric is given in E_n if, in all m-dimensional planes in E_n, the m-directions which correspond to points in $X_{m(n-m)}$ are measured by an ordinary Euclidean metric so that the translation of any E_m into another of the same direction is an isomorphic mapping of E_n. A Riemann metric in X_n is then defined by simply requiring that every tangent E_n has an m-dimensional Euclidean metric; from this one obtains an affine connection and the curvature tensor. By means of this metric the author states the variational problem for multiple integrals (in parametric form) and obtains the expression for stationary value (mean curvature $=0$). The equations of Jacobi are also obtained and they yield the second variation in invariant form. *M. S. Knebelman* (Pullman, Wash.).

Varga, O. Über eine Klasse von Finslerschen Räumen, die die nichteuklidischen verallgemeinern. Comment. Math. Helv. **19**, 367–380 (1947).

This note is based upon the tract by É. Cartan [Les espaces de Finsler, Actual. Sci. Ind., no. 79, Hermann, Paris, 1934]. The author characterizes Finsler spaces for which there is an absolute parallelism of the line-elements of support by the condition $R^k_{0hj}=0$. When this condition is satisfied there is defined a field of parallel line-elements $x'^i=x'^i(x^1, \cdots, x^n)$ in a whole region of the space. In this region therefore all the fundamental tensors of the Finsler space are functions of position. In particular, we can write $g_{ij}(x^1, \cdots, x^n, x'^1(x), \cdots, x'^n(x))=\mathfrak{g}_{ij}(x^1, \cdots, x^n)$. Expressing the fact that the $\mathfrak{g}_{ij}$ so obtained are to be the coefficients of a non-Euclidean space of curvature K, the author obtains the conditions $R^k_{ihj}=K(\delta^k_j g_{ih}-\delta^k_h g_{ij})$ as the conditions which, together with $R^k_{0hj}=0$, characterize the generalization of non-Euclidean spaces. He furthermore proves that, when the coefficients of affine connection in the Finsler space are independent of the line-element, the spaces just defined reduce to the well-known non-Euclidean spaces.

E. T. Davies (Southampton).

***Michal, Aristotle D. Matrix and Tensor Calculus with Applications to Mechanics, Elasticity, and Aeronautics.** John Wiley and Sons, Inc., New York; Chapman and Hall, Limited, London, 1947. xiii+132 pp. \$3.00. have

The purpose of this book is to give the reader a knowledge of the fundamentals of matrix calculus and tensor calculus and to show him how they can be applied in mathematics, physics, meteorology and engineering. The first part deals with the matrix calculus. Here and there a proof has been omitted but then the reader is referred to other literature. The applications lie in the fields of mechanical systems with a finite number of degrees of freedom (small oscillations, aircraft flutter). The second part contains an exact presentation of the tensor calculus, mainly for Euclidean spaces in general curvilinear coordinates. One chapter is devoted to Riemannian geometry. Besides the ordinary tensor field the multiple point tensor field (i.e., a field whose components depend on the coordinates of at least two points in space) is defined. The introduction of these fields appears to be useful in finite deformation problems. Several applications are given to hydrodynamics (equation of motion, boundary-layer theory) and to elasticity (finite deformation theory, strain tensor and stress tensor). *J. Haantjes.*

NUMERICAL AND GRAPHICAL METHODS

***Boll, Marcel. Tables Numériques Universelles des Laboratoires et Bureaux d'Étude.** Dunod, Paris, 1947. ii+882 pp. 3200 francs.

This is mainly a collection of about 300 numerical tables arranged in the following six groups: (A) Arithmetic and algebra, (T) Trigonometry, (E) Exponentials, (P) Probabilities, (C) Complex numbers, (U) Units, constants. In such a mass and variety of tables it is difficult to give any adequate idea of what is to be found in the volume, short of a complete listing such as is set forth on the large pages 873–882. Hence only some suggestions will be offered by noting a few types of tables which are to be found in each group.

In the first group are tables of $mn^{1/2}$ and $m/n^{1/2}$, $n=2, 3, 5, 10, 100$, $m=[1(1)100; 6D]$; values of series $\sum_{n=1}^{\infty} 1/n^p$, $p=[2(1)12; 10D]$, also as functions of powers of π; 20 tables (under one number A 27), for integral values 1(1)10 of A, B, N, D, of $(AB)^{1/2}$, $2AB/(A+B)$, $AB^{1/2}$, $AB^{1/3}$, $(A^2+B^2)^{1/2}$, $A/(A^2+B^2)^{1/2}$, $(A-1/B)^2$, N/D, N^2/D^2, N^3/D^2, $N/D^{1/2}$, $N^{1/2}/D$, N/D, $(N/D)^{1/2}$, $(N/D)^{1/3}$, $(N/D)^{1/4}$, $(N/D)^{1/5}$, $2^{1/2}N/D$, $3^{1/2}N/D$, $5^{1/2}N/D$; a molecular refraction table of $y=(x^2-1)/(x^2+2)$, $x=[1.251(.001)1.75; 6D]$; molecular polarization table of $y=(x-1)/(x+2)$, 6–8 D, x varying from 1.001 to 110, graphs being given for each of the last two functions. Graphs and reliefs to the number of 122 are a notable feature of the volume.

In the T group are tables of Bernoulli numbers, of values of the natural trigonometric functions and their multiples, of various combinations of trigonometric functions such as arise in the study of crystal structures, of elliptic, Fresnel, sine, cosine, and exponential integrals and of Legendre polynomials.

The E group includes various tables of financial mathematics, logarithms and antilogarithms, hyperbolic functions and functions of Planck, Einstein and Debye.

In the P group are tables of factorials and of Stirling's function, of gamma functions, of curves of Poisson, Gauss and Galton, of successive derivatives of the curve of Gauss, of Dirac's function and of probabilities of several variables.

The C group contains tables for the solution of algebraic cubic equations, for square roots of complex numbers, for expressing sin $(x\pm iy)$ and tan $(x\pm iy)$ in polar form, and inversely, and for Bessel functions (J_0, J_1 and their zeros; I_0, I_1, ber, bei, ber′ and bei′).

The U group includes such tables as of decimally expressed $n/365$ and $n/360$, for $n=1(1)365$ and 360; of radians per second corresponding to rotations per minute; of hydronamics, $v=(2gh)^{\frac{1}{2}}$; of expansion of a perfect gas (absolute temperature) $1+\alpha t$ and $1/(1+\alpha t)$, for $\alpha=1/273.15$; of spectroscopic units; of thermal radiance; of multiples of atomic masses, and of numerical chemical analyses; of horsepower and kilowatts.

It may thus be noted that the tables are intended to serve workers in many fields. They are presented in excellent outward form; for example, above most tables are definite references to pages where related formulae are given and notation explained. When graphs and reliefs occur there are also references to them. There is an extensive alphabetical index, pages 859–868.

Random checking of a few of the tables shows that serious errors are numerous. Some details in this regard are given in Math. Tables and Other Aids to Computation **2**, 337–338 (1947). We are told that about one third part of the tables was previously unpublished. *R. C. Archibald.*

Corrington, Murlan S. Table of the integral

$$\frac{2}{\pi}\int_0^x \frac{\tanh^{-1} t}{t}\,dt.$$

RCA Rev. **7**, 432–437 (1946).

This paper gives a table of the function

$$B(x)=\frac{1}{\pi}\int_0^x \log\left|\frac{1+t}{1-t}\right|\frac{dt}{t}$$

to 5 decimals for $x=0(0.01)0.97(0.05)0.99(0.02)1$. The function may also be expressed in the forms

$$B(x)=2\pi^{-1}\int_0^x t^{-1}\tanh^{-1} t\,dt=\pi^{-1}\{L(1+x)-L(1-x)\},\qquad 0\leqq t\leqq 1,$$

$$B(x)=\tfrac{1}{2}\pi-B(1/x)=\tfrac{1}{2}\pi-2\pi^{-1}\int_0^{1/x} t^{-1}\tanh^{-1} t\,dt$$

$$=\tfrac{1}{2}\pi-\pi^{-1}\{L(1+1/x)-L(1-1/x)\},\quad 1\leqq t<\infty.$$

In these expressions $L(u)=\int_1^u(u-1)^{-1}\log u\,du$ and is one of Spence's integrals. [See W. Spence, "An Essay on the Theory of the Various Orders of Logarithmic Transcendents," London-Edinburgh, 1809; 2d ed. 1820, in Spence's "Mathematical Essays"; F. W. Newman, "The Higher Trigonometry. Superrationals of the Second Order," Cambridge, England, 1892; E. O. Powell, Philos. Mag. (7) **34**, 600–607 (1943); these Rev. **5**, 110; all give tables of $L(u)$, denoted respectively by L^2, L and Rl, for various ranges of x; Newman gives also a table of $\phi(x)=\frac{1}{2}\{L(1+x)-L(1-x)\}$ to 12 decimals for $x=0(0.01)0.50$.] *J. C. P. Miller.*

Selmer, Ernst S. Some approximations for π. Norsk Mat. Tidsskr. **29**, 9–20 (1947). (Norwegian)

A new approximation to π. Math. Tables and Other Aids to Computation **2**, 245–248 (1947).

This note reports on two recent computations of π, one by D. F. Ferguson and the other the joint effort of L. B. Smith and J. W. Wrench. The latter result is given to 808 decimal places. Ferguson's 730 place value agrees with this and was found by the formula

$$\pi=12\cot^{-1}4+4\cot^{-1}20+4\cot^{-1}1985$$

and verified by the relation

$$\cot^{-1}20+\cot^{-1}1985=\cot^{-1}19.8.$$

Wrench and Smith used the famous Machin formula

$$\pi=16\cot^{-1}5-4\cot^{-1}239.$$

These results confirm Ferguson's earlier report [Nature **157**, 342 (1946); these Rev. **7**, 486] of the erroneous nature of Shanks's 1873 value beyond 527 decimal places and will require the revision of several notes on the distribution of the digits of π. One is relieved to find that the old deficiency of 7's is now comfortably made up. The actual sources of Shanks's erroneous calculations are discussed. All values of the above arccotangents of integers are given also. Unfortunately there is an error in arccot 5 so that the 730th through the 740th decimal of π should read 15981362977, not 11971602977. *D. H. Lehmer* (Berkeley, Calif.).

Birch, R. H. An algorithm for the construction of arctangent relations. J. London Math. Soc. **21**, 173–174 (1946).

The algorithm mentioned in the title consists in repeated use of the identity

$$\cot^{-1} a/b=\cot^{-1} c/d+\cot^{-1}(ac+bd)/(bc-ad).$$

To express $\pi/4$ as a sum of arccotangents for convenient computing one begins with $a=b=1$, chooses $d=\pm1$ and selects for c a sequence of powers of 10. Eleven steps of this process yield

$$\pi/4=8\cot^{-1}10-\cot^{-1}100-2\cot^{-1}1000$$
$$+\cot^{-1}(14717113539487181/715391356779).$$

Using the approximate formula $\cot^{-1}x=1/x$, the above formula gives $\pi/4$ correct to 13 decimals.

D. H. Lehmer (Berkeley, Calif.).

Salzer, H. E. The approximation of numbers as sums of reciprocals. Amer. Math. Monthly **54**, 135–142 (1947).

The paper studies chiefly the "R-expansion" of a number $s<1$ as a sum of reciprocals $s=1/a_1+1/a_2+\cdots$ such that $a_i\leqq 1/(s-s_{i-1})<a_i+1$. It has the properties: (a) it is unique, (b) it converges, (c) it terminates if s is rational, (d) the error of the ith partial sum is given by $s-s_i<1/(a_i^2-a_i)$, (e) $a_{i+1}=a_i^2-a_i+\epsilon_i$ with $\epsilon_i\geqq 1$ an integer, (f) $\epsilon_i>1$ for infinitely many i is necessary and sufficient that the series is the R-expansion of its limit. It follows that $\sum a_n^{-b_n^2}$, where a_n, b_n are nondecreasing sequences of integers greater than or equal to 2, cannot be a rational number. If $s_i=p/q$ is the ith approximation, the continued fraction has $|s-s_i|<1/q^2$, while the R-expansion has only $s-s_i<1/q$, but converges much more rapidly. A similar expansion, but with $|s-s_i|<1/2a_i(a_i-1)$ is obtained by choosing the a_i such that $|s-s_i|$ is as small as possible. An alternating series is obtained from $s=1/[s^{-1}]-r_1$, $r_1=1/[r_1^{-1}]-r_2$, $\cdots$. Here again $|s-s_i|<1/(a_i^2-a_i)$.

When comparing the R-expansion and the "ascending continued fraction" or "Teilbruchreihe" of Heis of about 1870 [called by the author "Pierce's method" of 1929]: $s=1/b_1\pm1/b_1b_2\pm1/b_1b_2b_3\pm\cdots$ with $b_{i+1}\geqq b_i$, the author overestimates the utility of the better convergence of the former. For as $a_{i+1}\approx a_i^2$, the a_i will soon have more figures than the calculating machine can take so that $1/a_k$ will have to be computed by a geometrical series. On the other hand the Teilbruchreihe would still be computable easily. So the best way in practice would seem to be to use the R-expansion until a_k has about 2 figures less than the calculating machine can take, and thenceforth apply the Teilbruchreihe.

E. Bodewig (The Hague).

Salzer, Herbert E. Alternative formulas for direct interpolation of a complex function tabulated along equidistant circular arcs. J. Math. Phys. Mass. Inst. Tech. **26**, 56–61 (1947).

It is readily seen that the expression

$$\varphi(z)=\left(\sum_{k=0}^{n-1}\frac{A_kf_k}{z-z_k}\right)\Big/\sum_{k=0}^{n-1}\frac{A_k}{z-z_k},$$

where $z_0, z_1, \cdots, z_{n-1}$ are any n distinct points in the z-plane, reduces to f_i when $z=z_i$, $i=0, 1, \cdots, n-1$. The rational fraction $\varphi(z)$ is therefore an interpolating function for $f(z)$ where $f_k=f(z_k)$ no matter what nonzero values are assigned to the A's. In particular, if

$$A_k=1/(z_k-z_0)\cdots(z_k-z_{k-1})(z_k-z_{k+1})\cdots(z_k-z_{n-1})$$

the expression $\varphi(z)$ reduces to Lagrange's interpolating polynomial. The author obtains formulas for calculating the A_k for the case in which the points $z_0, z_1, \cdots, z_{n-1}$ are equally spaced on the arc of a circle in the complex plane. The formulas are expressed in terms of a parameter θ, where θ is the constant angle between the chords joining successive pairs of points. All the A's are given for $n=3$ to $n=9$ inclusive. *W. E. Milne* (Corvallis, Ore.).

Michel, J. G. L. Central difference formulae obtained by means of operator expansions. J. Inst. Actuar. **72**, 470–480 (1946).

The author employs the fundamental identities

$$\delta=2\sinh\tfrac{1}{2}D,\quad \mu=2\cosh\tfrac{1}{2}D,$$

to develop formulas involving central differences, explains the underlying principles of the development and obtains in symbolic form central difference formulas for differential coefficients, subtabulation, summation, interpolation and quadrature. *W. E. Milne* (Corvallis, Ore.).

Moon, Parry, and Spencer, Domina Eberle. Analytic expressions in photometry and colorimetry. J. Math. Phys. Mass. Inst. Tech. **25**, 111–190 (1946).

The authors consider the fundamental equation in photometry and colorimetry, (1) $L=\int_0^\infty w(\lambda)J(\lambda)d\lambda$, where J represents the spectral distribution, w the weighting function depending on the receptor and L the effect on the receptor (such as photoelectric current). Since w is usually specified numerically by an agreed summary of empirical results, the evaluation of L, even for a mathematically specified spectrum J, would proceed by numerical quadrature. The authors, therefore, following their earlier line of approach, propose analytic approximations to w. Dealing specifically with the trichromatic weighting tables $\bar{x}(\lambda)$, $\bar{y}(\lambda)$, $\bar{z}(\lambda)$ (adopted in 1931 by the Commission Internationale de l'Eclairage) they fit (by least squares) an aggregate of Pearson type V curves, (2) $\sum_i A_i e^{-q_i/\lambda}\lambda^{-p_i}$. In the case of $\bar{z}(\lambda)$ one term in (2) is adequate whilst for $\bar{x}(\lambda)$ and $\bar{y}(\lambda)$ two or three terms are required over part of the range of λ. The fitted expressions are tabulated to an (at present unnecessary) accuracy of 8 significant figures, the range of λ being 0.360(.001).900 for $\bar{x}$ and $\bar{y}$ and 0.360(.001).660 for $\bar{z}$. For the general case of $w(\lambda)$ represented by a single Pearson type V term the formulas for L (in terms of A, p and q) have been listed (table III) for a number of well known analytic spectra $J(\lambda)$. Further tables and results deal with specific radiations. *H. O. Hartley* (London).

Kurdyumov, A. A. On the theory of ship hull design. C. R. (Doklady) Acad. Sci. URSS (N.S.) **55**, 91–94 (1947).

Variables x, y, z are rectangular coordinates in an axis system where x is measured lengthwise, y horizontally and z vertically downward; and $\xi=x/L$, $\eta=2y/B$, $\zeta=z/T$, where L, B, T are length, breadth and draught of the ship, respectively. Equation $\eta=f(\xi,\zeta)$ represents half the hull of a ship of given design. Let $0=f(\xi,\zeta)$ be expressed in the form $\zeta=g(\xi)$ (lower boundary of the half hull). Here $0\leqq g(\xi)\leqq 1$.

Given the values $\xi_0, \cdots, \xi_n, \zeta_0, \cdots, \zeta_m$ and the functions $S_n(\zeta)$ and $W_m(\xi)$, the author considers the problem of determining a new hull design in the form

$$\eta=h(\xi,\zeta)=f(\xi,\zeta)+\phi(\xi)\psi(\zeta)\{1-\zeta/g(\xi)\},$$

where ϕ and ψ are functions to be determined so as to satisfy the conditions

$$(1)\qquad \begin{cases} h(\xi_i,\zeta)=f(\xi_i,\zeta), & i=0,\cdots,n-1,\\ h(\xi,\zeta_j)=f(\xi,\zeta_j), & j=0,\cdots,m-1,\end{cases}$$

$$(2)\qquad h(\xi_n,\zeta)=S_n(\zeta),\quad h(\xi,\zeta_m)=W_m(\xi).$$

It is evident from (1) that ϕ and ψ must then satisfy the conditions $\phi(\xi_i)=0, i=0,\cdots,n-1; \psi(\zeta_j)=0, j=0,\cdots,m-1$. Since the author determines ϕ and ψ from two equations merely expressing the fact that (2) are satisfied, it seems to the reviewer that the author's "solution" does not satisfy (1).

The author then gives a "solution" of the problem where (2) are replaced by the equation $\int_0^{g(\xi)}h(\xi,\zeta)d\zeta=\omega(\xi)$, where $\omega(\xi)$ is a given function of ξ; and again seems to neglect (1). *A. B. Brown* (Flushing, N. Y.).

Jossa, Franco. Risoluzione progressiva di un sistema di equazioni lineari. Analogia con un problema meccanico. Rend. Accad. Sci. Fis. Mat. Napoli (4) **10**, 346–352 (1940).

The author solves the system of linear equations $\mathbf{Ax}=\mathbf{r}$ or $\mathbf{x}=\mathbf{A}^{-1}\mathbf{r}$ by splitting off from $\mathbf{A}$ the first principal subdeterminants, $\mathbf{A}_1=a_{11}, \mathbf{A}_2, \cdots, \mathbf{A}_{n-1}, \mathbf{A}_n=\mathbf{A}$ of orders $1, 2, \cdots, n-1, n$, and forming their inverses successively. Thus his method is, even in detail, the same as that of Boltz [Veröffentlichungen des Preussischen Geodätischen Instituts. N. F. no. **90**, Berlin, 1923], which again is the same as the method of computing the inverse of a matrix of matrices by the relation

$$\mathbf{D}^{-1}=\begin{pmatrix}\mathbf{P} & \mathbf{Q}\\ \mathbf{R} & \mathbf{S}\end{pmatrix}^{-1}=\begin{pmatrix}\mathbf{P}^{-1}+\mathbf{TU} & -\mathbf{T}\\ -\mathbf{S}_1^{-1}\mathbf{U} & \mathbf{S}_1^{-1}\end{pmatrix},$$

where $\mathbf{S}_1=\mathbf{S}-\mathbf{RP}^{-1}\mathbf{Q}$, $\mathbf{T}=\mathbf{P}^{-1}\mathbf{QS}_1^{-1}$, $\mathbf{U}=\mathbf{RP}^{-1}$. In the present case $\mathbf{D}=\mathbf{A}_{i+1}$, $\mathbf{P}=\mathbf{A}_i$, $\mathbf{S}=a_{i+1,i+1}$. [Cf. the review of a paper by Guttman, Ann. Math. Statistics **17**, 336–343 (1946); these Rev. **8**, 171.] *E. Bodewig* (The Hague).

Yates, Kenneth P., and Sinclair, George. A cam for introducing periodic functions into mechanical drives. Rev. Sci. Instruments **18**, 454–455 (1947).

In evaluating an integral of the form $\int f(\theta)g(\theta)d\theta$, one of the functions can be included in a cam of the shape $\phi=C_1\int g(\theta)d\theta+C_2$, so that the integral required becomes $C_1^{-1}\int f(\phi)d\phi$. The authors apply the method by modifying the paper drive of a strip-chart recorder so that the integral $\int_0^\pi P(\theta)\sin\theta d\theta$ is recorded as $(2/\pi)\int_0^\pi P(\phi)d\phi$ and hence can be evaluated by means of a planimeter. *S. H. Caldwell.*

Bergman, Stefan. Punch-card machine methods applied to the solution of the torsion problem. Quart. Appl. Math. **5**, 69–81 (1947).

The practical solution of boundary value problems usually leads to excessive calculation, a difficulty that

modern computational devices can overcome whenever the problem can be put in a form which these machines can handle. The present paper illustrates the application of orthogonal functions to the solution of Laplace's equation in two dimensions through the use of punch card machines. Essentially the process consists in (1) determining the integrals $F_{pq}=\iint_B z^p\bar{z}^q dxdy$ $(z=x+iy,\ \bar{z}=x-iy)$ for the region in question; (2) determining an orthonormal set of polynomials $\varphi_n(z)$ for the region; (3) determining the coefficients in an orthonormal series so that the real part of the series assumes the assigned boundary values; (4) determining the numerical values given by the real part of the series and its derivatives in the interior of the regions. The author shows how the punch card method may be applied to each of the foregoing operations and illustrates the procedure by giving the numerical solution of a torsion problem for a bar with an unusual cross section. *W. E. Milne* (Corvallis, Ore.).

Fox, L. Mixed boundary conditions in the relaxational treatment of biharmonic problems (plane strain or stress). Proc. Roy. Soc. London. Ser. A. **189**, 535–543 (1947).

Glagolev, A. A. A new method of nomographing a general nomographical equation of the third order. C. R. (Doklady) Acad. Sci. URSS (N.S.) **54**, 199–200 (1946).

A construction for an alignment chart of genus 2 for an equation of nomographic order 3 is given. The common carrier of two of the scales is a conic; the third scale is plotted on a straight line. J. Clark [Revue de Mécanique **21**, 321–335, 576–585 (1907)] showed that it is always possible to represent any equation of nomographic order 3 by a conic alignment chart [see also M. d'Ocagne, Calcul Graphique et Nomographie, O. Doin, Paris, 1908, p. 288]. The construction given by the author makes it possible to select convenient scales and carriers easily. *E. Lukacs.*

RELATIVITY

Narlikar, V. V. Gravitation. J. Univ. Bombay (N.S.) **8**, part 3, 70 pp. (1939).

This expository article contains a critical survey of the fundamentals of the general theory of relativity and of relativistic cosmology. A good deal of the author's philosophy of science is included. The topics discussed are as follows: Newtonian theory of gravitation; hypotheses and postulates in general relativity; Milne's kinematic relativity; the n-body problem by the approximation method of Einstein, Infeld and Hoffmann; nebulae and cosmology.
A. E. Schild (Princeton, N. J.).

Narlikar, V. V. The two-body problem in Einstein's new relativity. Proc. Nat. Inst. Sci. India **7**, 237–246 (1941).

The method of Einstein, Infeld, and Hoffmann [Ann. of Math. (2) **39**, 65–100 (1938)] is applied to the plane two-body problem and the equations of motion obtained to the second order of approximation [compare H. P. Robertson, Ann. of Math. (2) **39**, 101–104 (1938)]. The non-Newtonian terms are examined in some detail. *A. E. Schild.*

Narlikar, V. V. The consistency of Einstein's new relativity with the geodesic postulate. Current Sci. **10**, 164–165 (1941).

The relativistic equations of motion of two gravitating mass particles were obtained by an approximation procedure by Einstein, Infeld and Hoffmann [Ann. of Math. (2) **39**, 65–100 (1938)]. The author points out that, if one of the masses is zero, its equation of motion reduces to that of a geodesic, to the order of approximation considered.
A. E. Schild (Princeton, N. J.).

Narlikar, V. V., and Karmarkar, K. R. On a curious solution of relativistic field equations. Current Sci. **15**, 69 (1946).

The gravitational equations $G_{\mu\nu}=0$ are satisfied by the line element $ds^2=dt^2-(1+kt)^p dx^2-(1+kt)^q dy^2-(1+kt)^r dz^2$, where k is an arbitrary constant and p, q, r are constants subject to $p+q+r=2$, $pq+qr+rp=0$. This Riemannian space is in general not flat. *A. E. Schild.*

Narlikar, V. V., and Vaidya, P. C. The equations of fit in general relativity. Current Sci. **11**, 390–391 (1942).

The usual conditions imposed on the gravitational potentials for an isolated mass of fluid are that the $g_{\mu\nu}$ are continuous and that the pressure vanishes at the boundary of the fluid. The authors claim that these conditions are not always sufficient. A new boundary condition is proposed for the case of spherically symmetric distributions of fluid.
A. E. Schild (Princeton, N. J.).

Narlikar, V. V., Patwardhan, G. K., and Vaidya, P. C. Some new relativistic distributions of radial symmetry. Proc. Nat. Inst. Sci. India **9**, 229–236 (1943).

Particular solutions of the relativistic field equations are obtained for mass distributions with spherical symmetry. In addition to the usual requirements a new boundary condition is imposed [cf. the preceding review] which ensures that the total energy of the distribution equals the energy of the corresponding Schwarzschild mass particle.
A. E. Schild (Princeton, N. J.).

Vaidya, P. C. The external field of a radiating star in general relativity. Current Sci. **12**, 183 (1943).

The gravitational field of a radiating star is considered. A new relativistic line-element is proposed for the radiation zone. *A. E. Schild* (Princeton, N. J.).

Datta Majumdar, S. On the relativistic analogue of Earnshaw's theorem on the stability of a particle in a gravitational field. Bull. Calcutta Math. Soc. **38**, 85–92 (1946).

The problem of stability in relativity theory is discussed by means of geodesic deviation. It is shown that the equilibrium of a particle in a static gravitational field may be neutral to the first order or unstable, but can never be stable. This constitutes the relativistic analogue of Earnshaw's theorem of classical mechanics. *M. Wyman.*

Datta Majumdar, Sudhansu. A note on a class of solutions of Einstein's electro-static field equations. Science and Culture **12**, 295 (1946).

The author considers a line element of the form $ds^2=-e^u(dx_1^2+dx_2^2+dx_3^2)+e^w dt^2$, where u and w are independent of t. It is stated without proof that the field equations for empty space are solved by the expressions $u+w=0$,

$e^{\nu}=\frac{1}{4}(\varphi\pm\sqrt{2})^2$, where φ is the electrostatic potential. Detailed calculations are to appear in a later paper.
M. Wyman (Edmonton, Alta.).

Datta Majumder, Sudhansu. Note on a class of solutions of Einstein's field-equations in an electrostatic field. Science and Culture 12, 344 (1947).

It is stated without proof that an isotropic line element of a certain form is the only type compatible with a relationship given by the author. Complete calculations will appear later. *M. Wyman* (Edmonton, Alta.)

Dive, Pierre. Relations entre les potentiels de gravitation dans la matière en mouvement. C. R. Acad. Sci. Paris 224, 262–263 (1947).

This paper considers an electromagnetic field of perfect fluid type and derives conditions under which the covariant and contravariant forces will be central. *M. Wyman.*

Dive, Pierre. ds^2 d'univers dans la matière en mouvement. C. R. Acad. Sci. Paris 224, 633–635 (1947).

This paper derives the gravitational potentials of a spherically symmetric body in motion as functions of its density and velocity. *M. Wyman* (Edmonton, Alta.).

MECHANICS

Goldberg, Michael. Tubular linkages. J. Math. Phys. Mass. Inst. Tech. 26, 10–21 (1947).

A polyhedral linkage is a set of flat plates hinged along their common edges. It was proved by Cauchy [J. École Polytech., cahier 16, tome 9, 87–98 (1813)] that every closed convex polyhedral linkage is rigid. But Bricard [Leçons de Cinématique, vol. 2, Paris, 1927, pp. 7–12, 185–199, 311–322] described a closed nonconvex linkage that is continuously deformable. However, Bricard's polyhedron is singular, in that some faces necessarily interpenetrate. It remain an open question whether a deformable closed nonsingular polyhedron can exist. Some progress in that direction is made by these tubular linkages, which are built up from prismatic linkages each consisting of four or more quadrilateral plates (of ingeniously chosen shape) hinged in cyclic order along pairs of opposite sides. The paper is illustrated with photographs of models. Some of the linkages are of very remarkable appearance: e.g., one of the tubes is spiral, another helical. *H. S. M. Coxeter.*

Haag, Jules. Sur les joints homocinétiques. C. R. Acad Sci. Paris 224, 693–695 (1947).

The uniform rotation of a shaft A can be transmitted without variation to any other shaft B, whether B intersects A or not, by joining them through two Hooke's joints (universal joints) to an intermediate shaft C. The shaft C can be inserted in an infinity of ways subject to the condition that it makes the same angle with A as with B. The author was not aware of previous demonstrations of this theorem in texts by Reuleaux ["The Constructor," 1895, translated from the 4th German ed., p. 97] and Bricard [Leçons de Cinématique, v. 2, Paris, 1927, p. 190].
M. Goldberg (Washington, D. C.).

Pírko, Zdeněk. Sur le mouvement d'une figure plane variable. Časopis Pěst. Mat. Fys. 71, 71–77 (1946). (Czech. French summary)

Pour l'étude des propriétés géométriques d'une figure plane variable qui se meut dans son plan, on se sert habituellement de certaines relations dites "primordiales" ou "fondamentales" qui étaient déduites à son temps par MM. Mannheim et d'Ocagne. Ces relations ne sont qu'un cas très particulier des équations générales, que nous appelons "les équations généralisées de M. Cesàro pour l'analyse intrinsèque des courbes planes." La démonstration de ces équations et leur spécialisation à celles de Mannheim et de d'Ocagne font l'objet de notre travail.
Author's summary.

Kollros, Louis. Rotation d'un corps solide autour d'un axe. Elemente der Math. 2, 25–28 (1947).

Mangeron, D. I. Notes de mécanique rationnelle. Remarques sur le problème généralisant celui de la cloche et son battant. Bull. École Polytech. Jassy [Bul. Politehn. Gh. Asachi. Iaşi] 1, 304–308 (1946).

The equations of motion for a triple or n-tuple pendulum are written down and briefly discussed. *P. Franklin.*

Cassignol, Charles. Remarques à propos de l'entretien d'un pendule par le courant alternatif. C. R. Acad. Sci. Paris 224, 717–719 (1947).

The author considers a physical system in which a pendulum is maintained in vibration by an alternating current flowing in a circuit having an inductance which depends upon the instantaneous position of the pendulum. Some parts of the theory of the system are developed, on the basis of the assumption that the amplitude of vibration of the pendulum is small. In particular, there is a calculation of the minimum applied electromotive force which will maintain the vibration in the presence of dissipation.
L. A. MacColl (New York, N. Y.).

Vogel, Théodore. Les vibrations de certains systèmes couplés. Revue Sci. 84, 515–522 (1946).

The author discusses the effect of coupling two dynamical systems with infinitely many degrees of freedom and gives an example to illustrate his study. *A. E. Heins.*

Costa de Beauregard, Olivier. Sur la dynamique des systèmes de points. C. R. Acad. Sci. Paris 224, 333–334 (1947).

The author derives certain general equations pertaining to the dynamics of a system of point particles.
M. Wyman (Edmonton, Alta.).

Botez, Mihail Şt. Sur une surface cerclée particulière. Bull. École Polytech. Jassy [Bul. Politehn. Gh. Asachi. Iaşi] 1, 234–237 (1946).

Ghermănescu, studying the variation of the moment of momentum with respect to planes passing through a fixed straight line [Bull. Sci. École Polytech. Timişoara 8, 172–182 (1939)] has found that this variation is represented by the surface

$$F(x,y,z)=(x^2+y^2)^2(x^2+y^2+z^2)-\{x^2I(\lambda_1)+y^2I(\lambda_2)\}=0.$$

The author proves several properties of this surface. For example, besides its generating circles the surface contains six other circles lying in the planes symmetrical to the coordinate planes. *J. A. Schouten* (Epe).

Ghermănescu, Michel. Sur le mouvement tautochrone plan. Disquisit. Math. Phys. 1, 247–251 (1940).

The tautochronous trajectories are determined for a given central force field, with the force a function of polar distance.

Conversely, the law of central force is found for which a given trajectory is tautochronous. See the following review.
P. Franklin (Cambridge, Mass.).

Jacob, Caius. Observation concernant la note de Mr. M. Ghermănescu "Sur le mouvement tautochrone plan." Disquisit. Math. Phys. 1, 557 (1941).

The results of Ghermanescu's paper reviewed above were anticipated by Puiseux [J. Math. Pures Appl. (1) **9**, 409–421 (1844)]. *P. Franklin* (Cambridge, Mass.).

Spain, B. The generalised tautochrone. J. London Math. Soc. 21, 139–147 (1946).

The author considers a conservative dynamical system with N degrees of freedom. By application of $N-1$ workless constraints the motion in configuration space is confined to a curve C which passes through an equilibrium configuration O; C is a tautochrone if the periodic time of oscillation on C is independent of amplitude, or, equivalently, if the time of passage to O from rest at a point P on C is independent of the choice of P. The author finds that on a tautochrone the potential energy V must vary with arc length s ($ds^2=2Tdt^2$) according to the formula (*) $V=b+\mu s^2$, where b and μ are constants. [The reasoning is not clear to the reviewer and appears to be incorrect; an arbitrary parameter u is employed on C, but formulae leading to (*) are not invariant under transformation of the parameter. The formula (*) may be obtained easily by extending from 3 to N dimensions the argument of P. Appell, Précis de Mécanique Rationnelle, v. 1, Paris, 1919, p. 460.] The author discusses the relationship of tautochrones to quadratic systems of curves and, in particular, to geodesics. Since $d^3V/ds^3=0$ along a tautochrone, it follows that $P[V_{,mnp}]=0$ is the condition that all geodesics are tautochrones, the comma denoting covariant differentiation and P the sum of permutations. The paper concludes with a discussion of the case $N=2$; it is shown that, corresponding to each value of μ in (*), two tautochrones (possibly imaginary) pass through a point and form a harmonic pencil with the equipotential curve and its normal. *J. L. Synge.*

Laura, E. Sulla dinamica delle superficie flessibili ed inestendibili. Ist. Veneto Sci. Lett. Arti. Parte II. Cl. Sci. Mat. Nat. **100**, 671–687 (1941).

A general treatment of the small oscillations of a flexible inextensible surface based on the use of the moving trihedral, Weingarten's Verschiebungsfunktion and Beltrami's equations. As this general theory does not apply to surfaces of zero Gaussian curvature, a separate treatment is given along somewhat similar lines for developable surfaces.
D. C. Lewis (College Park, Md.).

Laura, E. Sul moto di una porzione di superficie conica inestendibile pesante. Rend. Sem. Mat. Univ. Padova 11, 113–131 (1940).

The general equations of Beltrami for the motion of a flexible and inextensible surface can be much simplified in case the surface is developable. The author considers the case when the developable surface is a cone. The method involves the kinematical aspects of a moving trihedral in terms of which the equations of Beltrami are properly interpreted. In this manner, "intrinsic equations" are obtained. More explicit results are obtained in case the cone can be unwrapped into the shape of a circular sector (the vertex corresponding to the center). Simplified equations for small oscillations about a position of stable equilibrium are also given. *D. C. Lewis* (College Park, Md.).

Agostinelli, Cataldo. Moto di due corpi rigidi pesanti collegati in un punto e di cui uno ha un punto fisso. Ist. Lombardo Sci. Lett. Cl. Sci. Mat. Nat. Rend. (3) **4(73)**, 663–678 (1940).

The motion under the influence of gravity of the system described in the title is studied as to the existence of first integrals. In general the only two integrals are the energy integral and the integral of the moment of momentum about the vertical line through the fixed point. In case the two bodies have gyroscopic structure there are two further first integrals and in this case there are special motions obtainable by quadratures. *D. C. Lewis* (College Park, Md.).

Agostinelli, C. Sui problemi dinamici con forze funzioni lineari delle velocità, per i quali esiste la funzione lagrangiana. I. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 1, 182–186 (1946).

It is shown that the system

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot q_r}\right)-\frac{\partial T}{\partial q_r}=\sum_{s=1}^{k}Q_{rs}\dot q_s+Q_{r0},\qquad r=1,\cdots,k,$$

can be reduced to

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_r}\right)-\frac{\partial L}{\partial q_r}=0,$$

where L has the form $L=T+\sum_{s=1}^{k}U_s\dot q_s+U$ if and only if $Q_{rs}=\partial U_s/\partial q_r-\partial U_r/\partial q_s$ and $Q_{r0}=\partial U/\partial q_r-\partial U_r/\partial t$. The author explains the relation of this result to a well-known result [Levi-Civita, Arch. Math. Naturvid. **31**, no. 12 (1911)] in the case of a particle activated by forces depending linearly on the velocity. *D. C. Lewis.*

Agostinelli, C. Sui problemi dinamici con forze funzioni lineari delle velocità per i quali esiste la funzione lagrangiana. II. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 1, 364–368 (1946).

[Cf. the preceding review.] A theorem is proved relative to the equivalence in certain instances of a system reduced by ignoration of certain coordinates to an unreduced system of fewer degrees of freedom and with a conservative field of force. *D. C. Lewis* (College Park, Md.).

Aymerich, G. Trasformazione non esattamente adiabatica ed integrazione approssimata di un particolare sistema canonico ad n gradi di libertà. Rend. Sem. Mat. Univ. Padova 12, 51–61 (1941).

Inequalities are obtained for the purpose of appraising the error in assuming that a dynamical system with Hamiltonian

$$H=\tfrac{1}{2}\sum_{i=1}^{n}(p_i^2+\omega_i^2(a)q_i^2)-\dot a\sum_{i,j=1}^{n}(b_{ij}(a)p_iq_j-\dot a c_{ij}(a)q_iq_j)$$

is adiabatic in the sense of Levi-Civita. Here a is the so-called adiabatic parameter, that is, a slowly varying function of the time t. The author uses a method, involving a certain canonical transformation, which he attributes to Graffi, who had previously treated the case $n=1$.
D. C. Lewis (College Park, Md.).

Pignedoli, Antonio. Estensione di un teorema di Joukowski al caso di un sistema olonomo, a vincoli indipendenti dal tempo e con n gradi di libertà. Atti Soc. Nat. Mat. Modena (6) 76, 94–102 (1945).

It is proved that, if the kinetic energy of a conservative holonomic dynamical system is $T=\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}\dot q_i\dot q_j$ and if the lines q_i=constant ($i=2, 3, \cdots, n$) are possible orbits,

then the potential energy U must satisfy the equations

$$\frac{a_{11}}{a_{i1}}\frac{\partial}{\partial q_1}\left(a_{i1}^2\frac{U+f}{a_{11}}\right)=f\frac{\partial a_{11}}{\partial q_i}+\frac{\partial}{\partial q_i}(a_{11}U), \quad i=2, 3, \cdots, n,$$

where f is an arbitrary function of $q_2, \cdots, q_n$. To solve for U, the a's must clearly satisfy certain integrability conditions. An application is given to the motion of a particle in space referred to elliptic coordinates. *D. C. Lewis.*

Pignedoli, Antonio. Sul moto di un corpo rigido pesante intorno ad un punto fisso prossimo al baricentro o poco differente dalla struttura giroscopica. Atti Soc. Nat. Mat. Modena (6) **76**, 115–143 (1945).

The general problem of the motion of a rigid body about a fixed point under the influence of gravity is considered from the point of view of first order variations from known special cases integrable by quadratures (elliptic functions). The following two problems are considered. First, the motions of Poinsot, in which the center of gravity is at the fixed point, are taken as the special integrable cases, while the variations are obtained by considering the distance of the center of gravity from the fixed point to be an infinitesimal. Second, the gyroscopic motions are taken as the special integrable cases and variations are obtained by considering as infinitesimal the difference between two of the principal moments of inertia and the distance of the center of gravity from one of the principal axes of inertia. Applications are made to questions of stability. *D. C. Lewis.*

Mattioli, Gian Domenico. Su di un principio variazionale centrale della dinamica. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **1**, 418–427 (1940).

The author formulates a general variational principle involving variation of time as well as of coordinates. The classical variational principles are derived as special cases. *D. C. Lewis* (College Park, Md.).

Arrighi, Gino. Sui sistemi anolonomi. Ist. Lombardo Sci. Lett. Cl. Sci. Mat. Nat. Rend. (3) **4(73)**, 367–374 (1940).

A discussion is given of the energy integral for conservative nonholonomic dynamical systems, the theory of which is well known. The significant part of the paper treats the analogous question of the exponential decay of energy in a rather restricted case of viscous friction. *D. C. Lewis.*

Arrighi, Gino. Sulla riduzione di rango dei sistemi alle caratteristiche pei moti inerziali. Ist. Lombardo Sci. Lett. Cl. Sci. Mat. Nat. Rend. (3) **4(73)**, 168–174 (1940).

This note treats the reduction of order of a system of differential equations corresponding to a dynamical system with null forces (but, in general, nonholonomic) when a first integral, linear in the velocities, is known. *D. C. Lewis* (College Park, Md.).

Marchetti, Luigi. Riduzione alla forma canonica delle equazione del moto di sistemi anolonomi. Ann. Scuola Norm. Super. Pisa (2) **10**, 199–208 (1941).

The differential equations of motion of a nonholonomic dynamical system can be written in the form

$$dp_i/dt=-\partial H/\partial q_i+\sum_j a_{ij}\lambda_j, \quad dq_i/dt=\partial H/\partial p_i,$$

$$\sum_i a_{ij}\dot{q}_i+a_j=0; \quad i=1, \cdots, n; j=1, \cdots, k.$$

These $2n+k$ equations involve $2n+k$ unknowns, namely, the p's, q's and λ's. The author obtains a necessary and sufficient condition that a transformation of the form

$$P_i=P_i(p, q, t), \quad Q_i=Q_i(p, q, t), \quad T=T(p, q, t)$$

shall have the effect of reducing the differential equations of motion to a system of $2n$ equations of the canonical form

$$dP_i/dT=-\partial K/\partial Q_i, \quad dQ_i/dT=\partial K/\partial P_i,$$

together with a system of k further relations among the P's, Q's and T. The condition is expressed in the form of a complicated system of $2n+k+1$ equations, involving as many unknown functions. Some special cases are discussed. Among these is the case in which the a_j's are zero, H and the a_{ij}'s are independent of t and K is equal to H. In the case in which the system is actually holonomic, the author's theory leads to the known results concerning contact transformations which preserve the canonical form of the equations of motion. *L. A. MacColl* (New York, N. Y.).

Vagner, V. Geometric interpretation of the motion of nonholonomic dynamical systems. Abh. Sem. Vektor- und Tensoranalysis [Trudy Sem. Vektor. Tenzor. Analizu] **5**, 301–327 (1941). (Russian) [MF 15618]

The author applies the differential geometry of nonholonomic varieties, more particularly the theory of V_3^2, to two problems of mechanics. One deals with a body which rests on a plane and may be considered as an idealized wheelbarrow: two supports are frictionless legs and one is a wheel with a sharp edge which, like an integraph wheel, can move only in one direction determined by its position. Translated into geometrical language the mechanical problem leads to a V_3^2 of zero curvature; this permits one to find a solution (which coincides with one given previously by Chaplygin). The other problem deals with a rigid body whose center of gravity is fixed; its possible positions are mapped on the points of a three-sphere made into a Riemannian space by taking the kinetic energy of the body as the fundamental form. The author finds that this space (which he calls the Euler space) is characterized by the property that the principal directions of the Ricci tensor are tangent to an orthogonal geodesic net and the characteristic roots are negative constants. A nonholonomic constraint is imposed on the body by fastening to it two diametrically opposite coplanar integraph wheels which are made to roll on the inner surface of a hollow sphere. The resulting motion may be interpreted as the motion in the Euler space (described above) of a point following an admissible curve of a nonholonomic V_3^2. With the help of this interpretation the problem is reduced first to the determination of a curve in ordinary space whose curvature and torsion are given by expressions involving inverse trigonometric functions, and finally to a second order linear differential equation of Riemann solved by hypergeometric functions. *G. Y. Rainich* (Ann Arbor, Mich.).

Tzortzes, A. Application of the theory of infinitesimal transformations to the study of a dynamical problem. Bull. Soc. Math. Grèce **22**, 191–194 (1946). (Greek)

Hydrodynamics, Aerodynamics, Acoustics

***Weinstein, A. Conformal representation and hydrodynamics.** Proc. First Canadian Math. Congress, Montreal, 1945, pp. 355–364. University of Toronto Press, Toronto, 1946. $3.25.

Expository lecture.

Cârstoiu, Ion. De la circulation dans un fluide visqueux incompressible. C. R. Acad. Sci. Paris **224**, 534–535 (1947).

The author observes that in a viscous incompressible fluid the existence of a potential for the accelerations implies the constancy (in time) of the circulation. *J. W. Calkin.*

Jacob, Caius. Recherches sur les mouvements fluides engendrés par la rotation de plusieurs corps solides. Disquisit. Math. Phys. **3**, 207–247 (1943).

This is the two-dimensional problem of a set of cylinders rotating as a rigid system about an axis, in an ideal incompressible fluid. General expressions are first deduced for the complex potential and for the force and moment on a cylinder in cyclic or acyclic flow, the fluid being bounded externally by a circular-cylindrical wall of radius R. Next, it is specified that the cylinders are n identical radial flat plates, symmetrically arranged, and $R\to\infty$. The cases $n=1$ and $n=2$ are first treated separately. Finally the same cases are considered with finite R. The method employed is, very briefly, that of solving the mixed Dirichlet problem for a half-plane by means of Green's functions; the technique used was given by A. Signorini [Ann. Mat. Pura Appl. (3) **25**, 253–274 (1916)]. *W. R. Sears* (Ithaca, N. Y.).

Jacob, Caius. Sur la seconde approximation dans le problème des jets gazeux. C. R. Acad. Sci. Paris **222**, 1427–1429 (1946).

The author calculates the second approximation of the problem of the gas jet according to the Janzen-Rayleigh method. In particular, the contraction coefficient is given in terms of that in the incompressible case. The calculation is done with the help of a formula attributed to M. E. Lamla [Luftfahrtforschung **19**, 358–362 (1943); these Rev. **5**, 19], which was also given by I. Imai and T. Aihara [see Imai, Proc. Phys.-Math. Soc. Japan (3) **24**, 120–129 (1942); these Rev. **7**, 344]. *C. C. Lin* (Cambridge, Mass.).

Truesdell, C. On Behrbohm and Pinl's linearization of the equation of two-dimensional steady polytropic flow of a compressible fluid. Proc. Nat. Acad. Sci. U. S. A. **32**, 289–293 (1946).

Behrbohm and Pinl have achieved the following linearization of the equation for the velocity potential [Z. Angew. Math. Mech. **21**, 193–203 (1941); these Rev. **7**, 495]:

$$(1)\qquad \omega_{\alpha\alpha}+\omega_{\beta\beta}+\frac{\omega_{\gamma\gamma}}{\lambda+(\mu-1)(\alpha^2+\beta^2)/\gamma^2}=0,$$

where $\omega(\alpha,\beta,\gamma)$ is homogeneous of degree 1 and α,β,γ are required to obey a subsidiary relation when transforming back to the flow plane; λ,μ are constants of the flow. The present paper shows that the same linearization can be obtained simply by means of the Legendre transformation, that the additional relation among the three variables is superfluous and, as a consequence, that α,β can be interpreted as the velocity components in the flow plane. It is shown also that the solutions of (1) obtained by separation of variables in cylindrical coordinates are essentially the Chaplygin solutions. *D. Gilbarg* (Bloomington, Ind.).

***Tables of Supersonic Flow Around Cones, by the Staff of the Computing Section, Center of Analysis, Under the Direction of Zdeněk Kopal.** Massachusetts Institute of Technology, Department of Electrical Engineering, Center of Analysis. Technical Report No. 1. Cambridge, Mass., 1947. xviii+558 pp. (10 plates).

Following Taylor and MacColl [Proc. Roy. Soc. London. Ser. A. **139**, 278–297, 298–311 (1933)], the authors of this volume seek "conical" solutions of the nonlinear differential equations of adiabatic flow, i.e., solutions for which the dependent variables are constant on coaxial cones passing through a common vertex. This will permit calculation of the flow about the nose of a conical projectile in symmetrical flight. The appropriate boundary conditions are given by (1) the condition at the surface of the projectile and (2) conditions at the conical shock wave that occurs at the nose. Solutions of the assumed type are found only when the shock wave is present, i.e., only for supersonic flight. The location of this wave is determined by means of the usual Rankine-Hugoniot relations, which are put into a convenient form for calculation.

The resulting set of equations, by virtue of their nonlinearity, must be solved numerically. For a given projectile nose angle θ_s it is most convenient to assume the value of the velocity u_s/c at the solid conical surface and to proceed outward until the shock wave is determined (c is the velocity corresponding to adiabatic expansion into a vacuum). The conditions upstream of the shock are the appropriate free-stream conditions. The well-known fact is pointed out, that this procedure actually leads to nonunique solutions: there are two shock-wave angles θ_w and two values of u_s/c for each stream Mach number M, but only one solution, the "first," is observed experimentally. The "first" and "second" solutions coincide at the minimum value of M for a given θ_s.

Part I of this volume consists of tables representing 268 integrations of the equations. Each case is characterized by values of θ_s, u_s, M and θ_w, and for each case the velocity components and velocity of sound in the air are tabulated as functions of the coordinate θ for values between θ_s and θ_w. The range of variables includes some "second" solutions, which are so designated. In part II are tabulated u_s/c, θ_w, M, the ratios of temperatures, pressures, and densities on the cone to those immediately behind the shock wave, the ratios of those immediately behind to those before the wave, and the drag coefficient of the cone. In parts III–VI are given further tabular data regarding the solutions, including some of doubtful physical significance, which may be of interest in extensions of the work.

In parts I–VI, the adiabatic exponent γ has been given the value 1.405 throughout. In parts VII and VIII some of the data of parts I and II are repeated for $\gamma=4/3$. Part IX gives the change of entropy and adiabatic constant across conical shock waves for both $\gamma=1.405$ and $4/3$. Finally, some of the results are shown graphically in a series of diagrams.

The calculations were carried out with ordinary computing machines. The numerical accuracy, indicated by the number of significant figures carried, appears to be more than would be justified for any immediate use of the tables. This point is discussed briefly. *W. R. Sears.*

Sears, W. R. A second note on compressible flow about bodies of revolution. Quart. Appl. Math. **5**, 89–91 (1947).

This is a correction to the author's previous note [same Quart. **4**, 191–193 (1946)] and the reviewer's review of that

note [these Rev. 8, 108]. The last word on this subject is now $U^{-1}\Delta u = \beta^{-2}F(\beta n)$, where Δu is the maximum superstream velocity, U the free stream velocity, $\beta^2 = 1 - M^2$, M the free stream Mach number and n the thickness ratio of the body. The function $F(n)$ is equal to the value of $\Delta u/U$ for incompressible flow over a body of thickness ratio n.

H. S. Tsien (Cambridge, Mass.).

Sears, William R. On projectiles of minimum wave drag. Quart. Appl. Math. 4, 361–366 (1947).

The expression derived by von Kármán for the wave drag of a slender body of revolution [Atti del V Convegno della "Fondazione Alessandro Volta," 1935, pp. 222–276] is applied to a projectile with pointed bow and stern, resulting in an expression for the wave resistance entirely analogous to that for the induced drag of a wing in the Prandtl lifting-line theory. The induced-drag analogy is pursued to obtain additional expressions for the wave resistance. Application of these is made to determine the shape of projectiles having minimum wave drag with prescribed length and volume, and also with prescribed length and diameter.

D. Gilbarg (Bloomington, Ind.).

Kaplan, Carl. Effect of compressibility at high subsonic velocities on the moment acting on an elliptic cylinder. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1218**, 39 pp. (5 plates) (1947).

The iteration method attributed to Ackeret is used; i.e., the stream function Ψ is calculated in the form of a series wherein the first term represents a rectilinear steady flow, the second term, Ψ_1, gives the usual Prandtl-Glauert approximation, etc. The problem here is to obtain an improvement of the Prandtl-Glauert result for the moment acting on a slender elliptic cylinder at a small angle of attack. It is found that a first-step improvement of the Prandtl-Glauert result (which states that the moment is proportional to $(1-M_1^2)^{-\frac{1}{2}}$, M_1 being the stream Mach number) requires calculation of the fourth term in Ψ, i.e., Ψ_3. In a previous paper [same Tech. Notes no. **1118** (1946); these Rev. **8**, 107] the author calculated Ψ_1 and Ψ_2. He therefore proceeds to evaluate Ψ_3 and the complete first-step improvement of the moment. He is also able to evaluate the second-step improvement of the Prandtl-Glauert formula for the lift, for comparison with the earlier paper.

The results are presented graphically for elliptic cylinders of thickness ratio 0.05 to 0.20. The graphs include the center-of-pressure location, which would be constant according to the Prandtl-Glauert approximation; it is found to move rearward as M_1 increases. The results of the present investigation are compared with the author's calculations of the lift and moment by the method of Poggi [Tech. Rep. Nat. Adv. Comm. Aeronaut., no. **671** (1939)]. The formulae are found to be in complete agreement so far as terms common to the two expansions are concerned.

W. R. Sears (Ithaca, N. Y.).

Van Driest, E. R. Streamlines for the subsonic flow of a compressible fluid past a sphere. J. Appl. Phys. **18**, 194–198 (1947).

Streamlines and pressure distributions are plotted for incompressible flow and for Mach number $M=0.5$, using the results, correct to terms in M^4, of Tamada [Proc. Phys.-Math. Soc. Japan (3) **21**, 743–752 (1939); these Rev. **1**, 185] and [independently] Kaplan [Tech. Notes Nat. Adv. Comm. Aeronaut., no. **762** (1940)]. *W. R. Sears.*

Göthert, B., and Kawalki, K. H. The calculation of compressible flows with local regions of supersonic velocity. Tech. Memos. Nat. Adv. Comm. Aeronaut., no. **1114**, 13 pp. (7 plates) (1947).

[Translation of Zentrale für Wissenschaftliches Berichtswesen der Luftfahrtforschung des Generalluftzeugmeisters, Forschungsber. **1794** (1943).] This paper can be divided into two parts. The first part is a description of the method of calculating mixed subsonic and supersonic flows proposed by the authors, with a numerical example. The second part is a general discussion of the field of such mixed flows. The authors' method of calculation consists of (1) starting with a given incompressible flow over a slender body and using the Glauert-Prandtl method to construct the flow far away from the body up to a constant velocity line with velocity less than sonic velocity, (2) using the difference equation to continue the field towards the body till the local velocity of sound is reached, (3) using the characteristic method to complete the flow field of supersonic velocity. The example given treats the flow over a two-dimensional symmetrical biconvex airfoil of 7.15 per cent thickness at Mach number 0.86, omits step (2) and uses the Glauert-Prandtl rule up to the sonic velocity line for simplicity. The result shows that the supersonic velocity along the central 40 per cent chord surface is almost constant, a phenomenon which is characteristic of symmetrical mixed flows. The discussion in the second part shows that, in order to obtain symmetrical mixed flows around convex profiles at high free stream Mach numbers, it is necessary to introduce additional source and sink distributions in the original incompressible flow so that the body in the incompressible flow has a concave central part. The possibility of instability of symmetrical mixed flow is also mentioned. *H. S. Tsien* (Cambridge, Mass.).

Frankl, F. On the problems of Chaplygin for mixed sub- and supersonic flows. Tech. Memos. Nat. Adv. Comm. Aeronaut., no. **1155**, 32 pp. (4 plates) (1947).

Translation of a paper in Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **9**, 121–143 (1945); these Rev. **7**, 496.

Bergman, Stefan, and Greenstone, Leonard. Numerical determination by use of special computational devices of an integral operator in the theory of compressible fluids. I. Determination of the coefficients of the integral operator by the use of punch card machines. J. Math. Phys. Mass. Inst. Tech. **26**, 1–9 (1947).

Bergman, Stefan. On supersonic and partially supersonic flows. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1096**, 85 pp. (1946).

This is a continuation of the author's researches on compressible flows [same Tech. Notes, nos. **972, 973** (1945); **1018** (1946); these Rev. **7**, 342, 343; **8**, 295] where a formula was developed and applied for the stream function of a possible two-dimensional steady rotational flow as the imaginary part of an arbitrary analytic function of a certain complex variable. In the present note these results are improved and completed. In addition an analogous formula is derived which represents a stream function of a possible supersonic flow in terms of two arbitrary functions of one real variable. Some instances are discussed in which flow patterns defined in two neighbouring parts of the plane can be combined into one flow pattern in the combined domain, thus leading in some instances to mixed (i.e., partially supersonic) flows. *L. M. Milne-Thomson* (Greenwich).

Shapiro, Ascher H., and Edelman, Gilbert M. Method of characteristics for two-dimensional supersonic flow—graphical and numerical procedures. J. Appl. Mech. 14, A-154–A-162 (1947).

Hasimoto, Zirō, and Sibaoka, Yosio. On the flow with circulation of a compressible fluid past a circular cylinder. Proc. Phys.-Math. Soc. Japan (3) **23**, 696–712 (1941).

This problem has been treated by Lamb [British Aeronaut. Res. Comm. Rep. and Memoranda no. **1156** (1928)] and Tamada [Rep. Aeronaut. Res. Inst. Tôkyô Imp. Univ., no. **205** (1941)], using the Janzen-Rayleigh and Poggi methods, respectively, and computing terms of order M^2. The present authors proceed to the "second approximation"; i.e., terms of order M^4. Velocity potential, maximum surface velocity, critical Mach number, and lift are calculated and compared with the first approximation and the incompressible-fluid case. (The angular displacement of the stagnation points from the flow direction is used as a parameter to specify the circulation.) *W. R. Sears* (Ithaca, N. Y.).

Hasimoto, Zirō, and Sibaoka, Yosio. On the flow with circulation of a compressible fluid past a circular cylinder. II. A supplementary note. Proc. Phys.-Math. Soc. Japan (3) **25**, 575–577 (1943).

Numerical errors in the paper reviewed above are corrected. *W. R. Sears* (Ithaca, N. Y.).

Tomotika, Susumu, Urano, Kaoru, and Hasimoto, Zirō. Further studies on the lift and moment of a circular arc aerofoil which touches the ground with its trailing edge. Proc. Phys.-Math. Soc. Japan (3) **23**, 713–724 (1941).

In a previous paper [same Proc. (3) **20**, 15–32 (1938)] Tomotika and Imai considered the plane case of a circular-arc airfoil at an angle of attack in an incompressible inviscid stream parallel to a plane boundary, when the trailing edge touches the boundary. From studies of his own and from the limited numerical examples in the earlier paper, A. E. Green [Proc. London Math. Soc. (2) **46**, 19–54 (1939); these Rev. **1**, 90] drew certain conclusions regarding the effect of the ground on lift of such airfoils. In the present paper additional numerical results are supplied to show that Green's conjecture is not generally true.

W. R. Sears (Ithaca, N. Y.).

Jones, Robert T. Subsonic flow over thin oblique airfoils at zero lift. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1340**, 13 pp. (7 plates) (1947).

Morris, Rosa M. The two-dimensional hydrodynamical theory of moving aerofoils. IV. Proc. Roy. Soc. London. Ser. A. **188**, 439–463 (1947).

The author, in earlier papers [same Proc. Ser. A. **161**, 406–419 (1937); **164**, 346–368 (1938); **172**, 213–230 (1939); these Rev. **1**, 90], has treated the two-dimensional potential motion produced by airfoils of a general class in arbitrary nonuniform motion. In the second of these papers it was stated that, upon reduction to the special case treated by von Kármán and Burgers [Aerodynamic Theory (W. F. Durand, editor), vol. 2, Springer, Berlin, 1934, pp. 280–315], there was disagreement in certain details. In the present paper some errors in the earlier work are corrected, principally in the interpretation of the general results, and it is shown that complete agreement with von Kármán and Burgers is obtained. Furthermore, in the case of harmonic oscillation of a thin airfoil, the results of von Kármán and Sears [J. Aeronaut. Sci. **5**, 379–390 (1938)] are easily obtained. The author also considers the stability of small oscillations. The theories of the other authors quoted here are based on the assumption that the wake vortices shed by the cylinder as the result of its varying circulation remain at rest in the fluid. In an appendix, the author considers this assumption in detail, calculating the actual velocity of a vortex near the cylinder. She concludes that there is some justification for the assumption for small angles of attack, although there is a first-order movement of the vortex sideways and a second-order drift downstream.

W. R. Sears (Ithaca, N. Y.).

Reissner, Eric. Effect of finite span on the airload distributions for oscillating wings. I. Aerodynamic theory of oscillating wings of finite span. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1194**, 39 pp. (1947).

A formula is derived for the pressure distribution on an oscillating lifting surface of finite span under the assumption of an aspect ratio that is not too small. The range of validity of this formula, so far as aspect-ratio limitations are concerned, is not less than the range of validity of lifting-line theory for the nonoscillating wing. It is found that the effect of three-dimensionality of the flow may be incorporated in the results of the two-dimensional theory by adding a correction factor to the basic function of the two-dimensional theory. The correction term is a function that depends on wing plan form, wing deflection function and the reduced frequency. Its determination requires the solution of an integral equation which is similar to the integral equation of lifting-line theory. The report concludes with an explicit statement of the form which the results of the theory assume for the spanwise variations of lift, total moment and hinge moments on a wing which is oscillating in bending, torsion and aileron and tab deflection.

From the author's summary.

Weissinger, J. The lift distribution of swept-back wings. Tech. Memos. Nat. Adv. Comm. Aeronaut., no. **1120**, 38 pp. (10 plates) (1947).

Translation of Zentrale für Wissenschaftliches Berichtswesen des Generalluftzeugmeisters, Forschungsber, no. **1553** (1942).

Kravtchenko, Julien. Sur l'existence des solutions du problème de Helmholtz dans le cas des obstacles possédant des points anguleux. Ann. Sci. École Norm. Sup. (3) **62**, 233–268 (1945).

Continuing previous investigations [J. Math. Pures Appl. (9) **20**, 35–303 (1941); these Rev. **3**, 219; **4**, 58], the author considers the wake due to a profile with angular points in a rectilinear channel. The existence of the wake is proved by topological methods for the solution of nonlinear functional equations. The author also refers to the solution of similar problems given previously by A. Weinstein [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) **4**, 119–123 (1926); **5**, 157–161 (1927)]. *A. Weinstein.*

Kantrowitz, Arthur. The formation and stability of normal shock waves in channel flows. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1225**, 41 pp. (10 plates) (1947).

The author extends the analysis of the celebrated Riemann memoir to the case of finite disturbances superimposed on the smooth flow in a duct of variable cross-section. In particular, he takes a smooth flow decelerating through the

speed of sound and considers the effect of small upstream moving disturbances, such as are likely to originate at the channel exit. It is shown that in the sonic portion of the channel the downstream reflections P which necessarily accompany an upstream disturbance Q are comparatively small and these are accordingly neglected in the first part of the paper. It is found that, on these assumptions, upstream moving pulses are "trapped" in the sonic portion at the throat and convert a portion of the flow to the steady accelerating flow appropriate to the channel. In the case of a compression pulse this portion is preceded by a shock (upstream of the throat), for expansion pulses it is followed by a shock (downstream of the throat).

This result is predicated on a shock velocity appropriate to the condition $P=0$. In part II of the paper, the author abandons the condition $P=0$ to the extent of calculating approximately the shock velocities corresponding to various realistic boundary conditions obtaining at the channel exit. In all cases the shock is shown to move upstream. Thus trapped expansion pulses are consumed by their own shock, while compression pulses build up indefinitely. The author adequately justifies the separation of the process into the "trapping" and the "consumption" phases. It thus emerges that smooth transonic decelerating flows are stable to expansions moving upstream from the exit, but are unstable to compressions.

The author then considers decelerating flows involving a shock. It is shown that of the two equilibrium shock positions, that in the diverging part of the duct is stable, that in the converging part unstable. It appears that a decelerating flow with a stable shock is stable to all except relatively powerful upstream moving disturbances.

D. P. Ling (Murray Hill, N. J.).

Martin, Monroe H. A problem in the propagation of shock. Quart. Appl. Math. 4, 330–348 (1947).

The author considers the following problem: given an infinitely long tube of gas at rest with the portion $|x|<1$ initially at uniform pressure p_0 and the portions $|x|>1$ at a lower uniform pressure p_2, what is the subsequent rectilinear flow? Following the Riemann procedure, a mapping of the "state plane" (r, s) (r and s are functions of the "state" variables μ, velocity, and ρ, density) on the space-time plane (x, t) is set up. The mapping is derived from a function $w(r, s)$ determined by a linear second order differential equation. In the case of a monatomic gas ($\gamma=5/3$) the general solution of this equation is relatively simple and the author for the most part confines himself to this case. Likewise, to avoid excessive analytic complexity, he requires p_0-p_2 to be small. The (x, t) plane divides up into a number of regions and the solution is followed from one to another by finding a new function w in each. Finally, the one-to-one character of the mapping breaks down, and certain regions of the (x, t) plane remain unexplored.

The following facts emerge: starting at the points $x=\pm1$, shock waves travel outward at a high constant velocity. (The shocks are assumed to be so weak that the entropy change across them is negligible.) At the same time two "buffer waves" travel inward toward $x=0$ where they react with each other. When the buffer waves and their reaction products meet the shock fronts, the shock velocities start to drop and eventually approach the speed of sound in the undisturbed gas. The history of the gas between the shocks is given in a series of graphs plotting ρ as a function of t for various values of x, and ρ as a function of x for various values of t, for the regions of the (x, t) plane reached by this analysis.

D. P. Ling (Murray Hill, N. J.).

Yadoff, Oleg. Sur le calcul des débits dans les écoulements permanents à la Poiseuille. C. R. Acad. Sci. Paris 224, 374–376 (1947).

L'auteur construit des exemples des écoulements permanents à la Poiseuille d'un fluide visqueux, incompressible à travers un tube à section droite rectangulaire. Il donne des tables numériques pour le calcul des vitesses le long des axes de symétrie du rectangle et indique une formule approchée très simple pour évaluer les débits.

J. Kravtchenko (Grenoble).

Hamel, Georg. Streifenmethode und Ähnlichkeitsbetrachtungen zur turbulenten Bewegung. Abhandlungen zur Hydrodynamik. XI. Abh. Preuss. Akad. Wiss. Math.-Nat. Kl. 1943, no. 8, 25 pp. (1944).

By applying the "strip-method" for the solution of differential equations, the author develops a theory of turbulence for two-dimensional flow through a channel which is substantially the same as von Kármán's similarity theory. Space average is used in dealing with turbulent fluctuations. The conditions near the wall and the center of the channel are treated more carefully. The difference between the laminar and turbulent velocity profiles near the wall is shown to vanish at least as fast as the fifth power of the distance from the wall.

C. C. Lin (Cambridge, Mass.).

de Marchi, Giulio. Profili longitudinali della superficie libera delle correnti permanenti lineari con portata progressivamente crescente o progressivamente decrescente entro canali di sezione costante. Ricerca Sci. 17, 202–208 (1947).

Stockmann, W. B. Equations for a field of total flow induced by the wind in a non-homogeneous sea. C. R. (Doklady) Acad. Sci. URSS (N.S.) 54, 403–406 (1946).

The equations of motion without vertical terms or acceleration terms, but containing horizontal and vertical friction, are combined with the equation of continuity (neglecting vertical divergence) into a single partial differential equation of the fourth order. The vertical coefficient of eddy viscosity is allowed to vary with height, but the horizontal coefficient is held constant. The dependent variable in this equation is the stream function of velocity transport; the independent variables are the horizontal coordinates. The surface stress of the wind is assumed given. A possible solution of the differential equation is indicated; it is a function of the Fourier components of the vorticity of the wind stress.

H. Panofsky (New York, N. Y.).

Stockman, W. B. A theory of T–S curves as a method for studying the mixing of water masses in the sea. J. Marine Research 6, 1–24 (1946).

The equations of vertical diffusion and vertical heat conduction are integrated for the case of three distinct horizontal water masses with different salinities and temperatures, subject to continuity of heat and salinity. The resulting salinity-temperature distribution is given graphically and several of its properties are derived. It is shown that the procedure can be inverted to yield the temperature and salinity of the central water mass when the S–T curve of the mixture is given. T. P. Jacobsen [Beitr. Geophys. 16, 404–412 (1927)] had shown that the vertical coefficient of eddy exchange may be found from successive S–T curves

by a simple geometrical procedure. Stockman shows that this procedure is not valid in all cases and devises a more rigorous procedure. *H. Panofsky* (New York, N. Y.).

Fjeldstad, J. E. Stationary currents in heterogeneous water. Arch. Math. Naturvid. **48**, no. 6, 157–175 (1946).

The equations of motion are integrated for a simple model: stationary currents in an infinitely long canal in the x-direction with no gradients of speed, pressure or density in that direction. Starting with known distributions of surface stress, the distribution of current speed with depth is derived and, from this, the distribution of pressure and density. With simple assumptions for the variation of stress with the direction at right angles to the current, the variation of current speed with depth becomes simpler than observed; also the slope of the isopycnals increases with depth contrary to observations. Starting from a more realistic distribution of current speed, the field of mass computed agrees well with observational experience.

H. Panofsky (New York, N. Y.).

Gridel, Henri. Essai d'application des résultats de la physique ondulatoire à l'étude des phénomènes de propagation de la houle. Ann. Ponts Chaussées **1946** (116e année), 77–105, 330–351 (1946).

The object of this paper is to study surface waves in water of variable depth and in regions with irregular boundaries, such as bays, harbors, dock areas, etc. The methods employed are essentially graphical and geometrical in character and depend basically upon the use of Huygens' principle in connection with the law relating the wave length, depth of water and propagation speed of a surface wave of small amplitude. A considerable number of special cases are worked out in detail, for example: refraction of waves around a corner, propagation of waves through a narrow opening in a breakwater, a study of standing waves in a harbor. The wave amplitudes as well as their geometrical patterns are estimated. *J. J. Stoker* (New York, N. Y.).

Hylleraas, Egil A. On the theory of tidal oscillations in oceans with solid boundaries. Geofys. Publ. Norske Vid.-Akad. Oslo **13**, no. 10, 12 pp. (1943).

The author formulates the problem of the small oscillations of an ocean on a rotating sphere under the influence of tidal forces as a variational problem whose "natural" boundary conditions are the actual physical ones. He then shows under certain assumptions that, for sea water which is stratified (i.e., with a moderate density gradient), the problem is approximately equivalent to one which corresponds to the two-dimensional tidal theory of Laplace, in contradistinction to the case of homogeneous sea water. This is supplementary to an earlier paper [Astrophys. Norvegica **3**, 139–164 (1939); these Rev. **1**, 90], where the same approximate variational formulation was arrived at by different arguments. The solution of the variational problem for the case of a flat sea bottom is discussed and the procedure to be followed for the case of variable depth briefly indicated. *J. W. Calkin* (Houston, Tex.).

Hylleraas, Egil A., und Romberg, Werner. Über die Schwingungen eines stabil geschichteten, durch Meridiane begrenzten Meeres. II. Berechnung der Eigenfrequenzen. Astrophys. Norvegica **3**, 247–271 (1941).

Part I, by Hylleraas, appeared in the same vol., 139–164 (1939); these Rev. **1**, 90.

Bordoni, P. G., e Gross, W. Massa di radiazione di un diaframma rigido munito di schermo acustico chiuso. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **1**, 395–401 (1946).

The aim of this paper is to obtain an expression for the acoustic inertial mass of a vibrating spherical sector acting as a source of sound waves. This problem leads to the Neumann problem for a cylinder for the wave equation $(\Delta+k^2)u=0$. An approximate solution has been obtained by Bordoni, where the inertial mass for such an acoustic system is given by

$$\text{(a)} \qquad M_0=a_1^3\rho_0 2\pi(1-x^2)^{-1}\sum_{n=0}^{\infty}\frac{2n+1}{2n+2}Q_n^2(x),$$

where ρ_0 is the density of the surrounding fluid, a_1 is the semi-chord of the sector, $x=\cos\theta$ is the distance from the center of the sphere to the sector and

$$Q_n(x)=\int_0^\theta P_n(\cos\theta)\sin\theta\cos\theta d\theta.$$

The authors have succeeded in calculating the infinite series in (a) by the aid of the generating function for Legendre polynomials. Two complex functions $F(x,u)$ and $G(x,u)$ are defined as follows:

$$F(x,u)=\int_u^1 \xi f(\xi,u)d\xi=\sum_{n=0}^{\infty}Q_n(x)u^n,$$

$$G(x,u)=u^{-1}\int_0^u F(x,\lambda)d\lambda=\sum_{n=0}^{\infty}Q_n(x)\frac{u^n}{n+1}.$$

By integrating FF^* and FG^* around a unit circle the author obtains two functions $Z_1(x)$ and $Z_2(x)$ whose difference gives precisely the series (a); it is a finite sum of sines and cosines and an infinite series in θ whose coefficients contain Bernoulli numbers. The series is found to converge for $|\theta|<\pi$. For the case when the radius of the sphere becomes infinite, expression (a) reduces to Rayleigh's formula $M_0=8a_1^3\rho_0/3$ [Theory of Sound, 2d ed., v. 2, London, 1896, p. 164, equation (14)] for an oscillatory rigid disk of radius a_1 embedded in an infinite rigid wall. The ratio of $M_0/a_1^3\rho_0$ for the flat disk to that for the spherical sector is tabulated for θ varying from 0 to 90° and plotted. The inertial mass decreases as θ is increased from 0 to 90°. The maximum value of M_0 is when $\theta=0$ (flat circular disk) as given by Rayleigh. *N. Chako* (Manhattan, Kan.).

Bordoni, Piero Giorgio. Sul moto di una membrana elastica accoppiata ad un sistema acustico. Ricerca Sci. **13**, 820–827 (1942).

This paper deals with the problem of finding the conditions for which the average amplitude of vibration of a membrane, subjected to a simple harmonic force, is independent of the impressed frequency. Here the reaction pressure of the acoustic system is taken into account. The latter is found by multiplying the acoustical impedance by the average velocity of the membrane. The average displacement η_0 is expressed in terms of Bessel functions,

$$|\eta_0|=|\omega^2\rho J_0(z)/J_2(z)+i\omega Z_a|, \qquad z=\tfrac{1}{2}\omega a/v,$$

where $Z_a=R_a+iX_a$ is the acoustic impedance of the system. The author introduces a function of the frequency $f(\omega)$ (the phase function) and expresses R_a and X_a and the acoustic rigidity $S_a=-\omega X_a$ in terms of $f(\omega)$. Assuming a maximum value of η_0, which is the case for $f(\omega)=0$, $\omega=0$ (static deflection under a constant tension T), the author

calculates the acoustic rigidity; this case cannot be realized in practice. The case $f(\omega)=$constant is likewise unattainable physically. The author then considers the case

$$f(\omega)=\tan^{-1}\{kz/(z_0^2-z^2)\}$$

(k is a parameter and z_0 is a root of $J_0(z)=0$) and calculates R_a and S_a as functions of z. To determine an appropriate value of k one considers the corresponding acoustical problem of the membrane vibrating against a plate of radius a, placed at a distance d from it, for which $R_a'(a,d)$ and $S_a'(a,d)$ are known [Crandall, Physical Rev. (2) 11, 449–460 (1918)]. By varying a and d in such a way that R_a' and S_a' have points in common with R_a and S_a, these points then will make the difference for both functions R_a and S_a a minimum in a certain range of frequencies.

N. Chako (Manhattan, Kan.).

Primakoff, Henry, Klein, Martin J., Keller, Joseph B., and Carstensen, E. L. Diffraction of sound around a circular disk. J. Acoust. Soc. Amer. 19, 132–142 (1947).

The sound field behind a circular disk in a given incident sound field is computed approximately by a calculation based on the Kirchhoff surface integral. This surface integral may be reduced to a line integral by the Maggi transformation [Baker and Copson, "The Mathematical Theory of Huygens' Principle," Oxford University Press, 1939, p. 23; these Rev. 1, 315]. This calculation is compared with a more accurate calculation which makes use of the Green's function.

A. E. Heins (Pittsburgh, Pa.).

Haskind, M. D. Acoustical radiation of oscillating bodies in a compressed liquid. Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. 16, 634–646 (1946). (Russian. English summary)

Elasticity, Plasticity

Locatelli, Piero. Sulla congruenza delle deformazioni. Ist. Lombardo Sci. Lett. Cl. Sci. Mat. Nat. Rend. (3) 4(73), 457–464 (1940).

In a previous note [Boll. Un. Mat. Ital. (2) 2, 342–347 (1940); these Rev. 2, 172] the author obtained the equations of compatibility from the principle of minimum energy. The aim of the present paper is to obtain the equations of compatibility (1) for an elastic continuum from the principle of virtual work and (2) for a quasielastic continuum from a principle called the stationary character of the second energy of deformation. In (1), the argument is formulated for a curved membrane of constant curvature and for a three-dimensional Euclidean continuum, with extension to the case of a three-dimensional continuum with constant curvature. For this review, only the case of the Euclidean 3-space will be discussed. Following B. Finzi [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) 19, 378–382 (1934)], the author takes a stress tensor of the form (*) $\Phi^{ij}=\epsilon^{ihr}\epsilon^{jks}\chi_{rs|hk}$, where the ϵ's are permutation symbols, the stroke indicates partial differentiation and χ_{rs} is any symmetric tensor. Any such stress distribution automatically satisfies the equations of equilibrium in the absence of body forces. Conditions are imposed on χ_{rs} to make the surface stress vanish. The principle of virtual work is taken to be (**) $\int\xi_{ij}\Phi^{ij}dV=0$, where ξ_{ij} is a symmetric deformation tensor. Substitution from (*), with use of Green's theorem and the boundary conditions, gives

$$\int\xi_{ij|hk}\epsilon^{ihr}\epsilon^{jks}\chi_{rs}dV=0.$$

The χ_{rs} are regarded as arbitrary (except for boundary conditions), and hence we have $\epsilon^{ihr}\epsilon^{jks}\xi_{ij|hk}=0$, which are in fact the equations of compatibility, implying the existence of a vector s_i such that $2\xi_{ij}=s_{i|j}+s_{j|i}$. Thus the equations of compatibility on ξ_{ij} may be regarded as the conditions of orthogonality (in the sense of (**)) of a deformation ξ_{ij} to all stress distributions Φ^{ij} satisfying the equations of equilibrium under no body forces and giving zero stress across the boundary. As regards (2), the author takes for second energy of deformation $F=-\int\{p^{ik}\xi_{ik}-\int p^{ik}d\xi_{ik}\}dV$, where p^{ik} is stress and ξ_{ik} deformation. The stationary principle $\delta F=0$ gives formally $\int\xi_{ik}\delta p^{ik}=0$. If the variations δp^{ik} are subject to the equations of equilibrium and the condition of vanishing surface stress, this last equation is of the same form as (**), and the same deduction may be drawn, viz. that ξ_{ik} satisfies the equations of compatibility.

J. L. Synge (Pittsburgh, Pa.).

Murnaghan, F. D. A revision of the theory of elasticity. Bol. Soc. Mat. Mexicana 2, 81–89 (1945).

Lecture to the Mexican National Mathematical Congress.

***Sokolovskiĭ, V. V. Teoriya Plastičnosti. [Theory of Plasticity].** Izdatel'stvo Akademii Nauk SSSR, Moscow-Leningrad, 1946. 306 pp.

By far the greater part of this book is concerned with the classical theory of plasticity [Saint Venant, Lévy, von Mises]. As far as this particular theory of plasticity is concerned, the present book doubtless constitutes the most complete treatment available to date. Unfortunately, other theories of equal practical importance or mathematical interest are barely mentioned, or not at all, so that the uninitiated reader will receive the impression that the classical theory is the only one that matters. The basic equations of the classical theory are developed in the first two chapters which also contain some remarks concerning mechanical similarity in the plastic range. The wide range of topics treated in the remaining ten chapters is best illustrated by the following list: problems with cylindrical or spherical symmetry; torsion of cylindrical and prismatic bars and of bodies of revolution; problems of plane strain (plastic stress distributions in the neighborhood of cut-outs, impression of a rigid wedge into a plastic body, rolling and drawing); problems of plane stress (contrary to what is the case for plane strain, it makes a difference here whether the yield condition of Saint Venant or that of von Mises is used); bending of beams and plates. Throughout the book, the author stresses methods of approximate numerical integration which permit the handling of much more complicated boundary conditions than are accessible to analytical methods.

W. Prager (Providence, R. I.).

Bijlaard, P. P. On the restricted applicability of the principle of least work in the plastic domain. Nederl. Akad. Wetensch., Proc. 50, 397–405 (1947).

The mechanical behavior of the solid considered in this paper can be interpreted as arising from the superposition of the deformations of (1) a compressible elastic material and (2) an incompressible plastic material which obeys stress-strain relations of the deformation type [Hencky-Nadai], both materials being subjected to the same stress. The author shows at length that the principle of minimum strain energy does not apply to such a material. [This is

not surprising because this principle applies only to the elastic constituent, whereas the behavior of the plastic constituent is regulated by the principle of minimum complementary energy.] *W. Prager* (Providence, R. I.).

Kachanov, L. M. On the stress-strain relations in the theory of plasticity. C. R. (Doklady) Acad. Sci. URSS (N.S.) **54**, 309–310 (1946).

In the discussion of the mechanical behavior of plastic materials with strain-hardening it is customary to give two sets of stress-strain relations, one for loading, the other for unloading. The author points out that these may be combined by using the absolute value of the increment of the strain intensity. *W. Prager* (Providence, R. I.).

Ševčenko, K. N. The plastic stressed state and the flow of metals in cold rolling and drawing. Bull. Acad. Sci. URSS. Cl. Sci. Tech. [Izvestia Akad. Nauk SSSR] **1946**, 329–354 (1946). (Russian)

Using the theory of plasticity of Saint Venant and von Mises, the author discusses the distribution of stresses and velocities in a sheet which is drawn between two idler rolls. He assumes first that there is no friction between rolls and sheet and then that the frictional stress is proportional to the relative speed of the surface of the roll with respect to the sheet material. In both cases the construction of the net of slip lines (characteristics) is discussed at length. A numerical example is given. *W. Prager.*

Milne-Thomson, L. M. Stress in an infinite half-plane. Proc. Cambridge Philos. Soc. **43**, 287–288 (1947).

The author solves the problems of plane strain and generalized plane stress in an isotropic elastic half-plane with prescribed distribution of stresses at the straight edge. [This problem (as well as the problem involving a prescribed distribution of displacements) was exhaustively treated in N. I. Muschelišvili's book, On Certain Basic Problems of Mathematical Theory of Elasticity, Acad. Sci. USSR, 2d ed., 1934, pp. 318–341. See also G. V. Kolossoff, Z. Math. Phys. **62**, 384–409 (1914); M. Sadowsky, Z. Angew. Math. Mech. **8**, 107–121 (1928); **10**, 71–81 (1930).]
I. S. Sokolnikoff (Los Angeles, Calif.).

Dean, W. R., and Wilson, A. H. A note on the theory of dislocation in metals. Proc. Cambridge Philos. Soc. **43**, 205–212 (1947).

With a view to discussing Bragg's dislocation theory, the author studies the following problem in plane strain: in an otherwise infinite elastic continuum two cylindrical holes of equal radii R and parallel axes are connected by a cut, the plane of which contains the axes of the holes. The banks of the cut are made to slip over each other by an amount s and are then joined together. The resulting system of internal stresses is analyzed and the strain energy in a slice of unit thickness is evaluated. The latter depends, of course, on the amount s of the dislocation, the distance t between the axes of the holes, and, to a lesser extent, on the radius R. In Bragg's theory, t is the average distance between faults and s the distance between atoms in the slip plane, while R is introduced here to exclude a region of rapid transition in which the physical conditions are not well known, but where the energy density cannot exceed a certain natural limit. Comparing his theoretical value of the energy density with that found in cold-worked copper by Taylor and Quinney [Proc. Roy. Soc. London. Ser. A. **143**, 307–326 (1934)], the author arrives at the estimate of 0.9×10^7 dislocations per centimeter on an interface between two mosaic elements. This agrees well with Bragg's estimate of 1.5×10^7 dislocations per centimeter. *W. Prager.*

Bowie, O. L. Elastic stresses due to a semi-infinite band of hydrostatic pressure acting over a cylindrical hole in an infinite solid. Quart. Appl. Math. **5**, 100–101 (1947).

Bažant, Zdeněk. Theorie des elastischen Halbraumes. Acad. Tchèque Sci. Bull. Int. Cl. Sci. Math. Nat. **44**, 313–329 (1943).

The determination of stresses and displacements in an elastically isotropic half-space was made by Boussinesq with the aid of logarithmic potentials. In this paper the author considers several special cases of Boussinesq's problem by making use of Airy's stress function rather than logarithmic potentials. *I. S. Sokolnikoff* (Los Angeles, Calif.).

Volterra, E. Problemi dinamici della trave in regime ereditario. I. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **2**, 42–47 (1947).

Volterra, E. Problemi dinamici della trave in regime ereditario. II. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **2**, 178–180 (1947).

The author considers a visco-elastic material for which the tensile stress σ at the instant t and the corresponding unit elongation ϵ are related by means of

$$\sigma(t)=E\epsilon(t)+\int_0^t\Phi(\tau)\epsilon'(t-\tau)d\tau,$$

where the prime indicates differentiation with respect to the time and $\Phi(t)=A\exp(-at)$, A and a being constants. The equation for the transverse vibrations of a simply supported beam of this material is established and the following cases are treated in detail: free vibrations, forced vibrations caused by a load distribution of the form $p_0\sin(\pi x/l)\sin 2\pi nt$ (l, span of beam), force of constant intensity or pulsating force moving with constant velocity along the beam.
W. Prager (Providence, R. I.).

Weinstein, A. The center of shear and the center of twist. Quart. Appl. Math. **5**, 97–99 (1947).

Trefftz has utilized the reciprocity theorems of Maxwell and Betti to establish the center of shear in a cantilever beam of uniform section. His derivation shows that the shear center depends only on the shape of the cross section and is readily determined if the warping function of the corresponding Saint Venant torsion problem is known. If the root section of the cantilever beam is rigidly clamped, the torsion problem is not an exact solution in the usual sense and the displacements in the flexure and torsion problem of Saint Venant are insufficient to determine the center of twist or the point at rest in every normal section. Weinstein adopts an idea of Cicala for an approximate clamping by requiring that, on the root section $z=0$, the displacements u and v are identically zero and for every section the axial displacements satisfy $\iint w^2dxdy=\text{minimum}$. These requirements yield the same point for the center of twist as Trefftz obtained for the center of shear. Both points are independent of Poisson's ratio. It is also shown that the average value of w over any section is zero.
D. L. Holl (Ames, Iowa).

Weinstein, A., and Jenkins, J. A. On a boundary value problem for a clamped plate. Trans. Roy. Soc. Canada. Sect. III. (3) **40**, 59–67 (1946).

Weinstein has observed that the problem of statically loaded clamped thin plates can be interpreted as the limiting case of a sequence of problems starting from a base problem. The latter is readily solved from the equation of a membrane and the subsequent set of solutions are formulated as variational problems with an increasing number of subsidiary conditions. The authors show the existence and uniqueness of the solution for a region S which is bounded by arcs with continuously turning tangents except at a finite number of corners. It is furthermore shown that the sequence of solutions $w_m(x, y)$ of the intermediate set converges uniformly to the limit solution. This is also true of their derivatives if the loading function is of integrable square over S. The proofs are based on the use of a complete set of functions harmonic in S and orthogonal to $\nabla^2 w$, where w is the deflection of the plate. *D. L. Holl.*

Federhofer, Karl, und Egger, Hans. Berechnung der dünnen Kreisplatte mit grosser Ausbiegung. Akad. Wiss. Wien, S.-B. IIa. **155**, 15–43 (1946).

The authors study the system of nonlinear differential equations of von Kármán for large deflections of thin circular plates. The loading and edge conditions have radial symmetry. The clamped edge may be fixed or free to move radially. The use of power series solutions leads to nonlinear infinite systems for determining the coefficients in the series. The authors choose an approximate deflection function satisfying the boundary condition and determine the parameters in the membrane stress function by Galerkin's approximation method. This leads to a finite system of nonlinear algebraic equations for the desired parameters. Graphical and tabular data for the deflection, the membrane and flexural stresses are provided. *D. L. Holl* (Ames, Iowa).

Reissner, Eric. On bending of elastic plates. Quart. Appl. Math. **5**, 55–68 (1947).

The author rederives his theory of the bending of thin elastic plates [J. Math. Phys. Mass. Inst. Tech. **23**, 184–191 (1944); J. Appl. Mech. **12**, A-69–A-77 (1945); these Rev. **6**, 195; **7**, 42] in a more general form and presents a number of new applications of the theory. The author's general purpose is to improve the accuracy of the standard theory. In particular, his theory makes it possible to satisfy the physical conditions at a free edge of a plate much more accurately than was possible with the classical theory, without at the same time introducing serious mathematical difficulties. One of the new problems treated is that of a "sandwich plate," consisting of a plate built up of three layers of material, the one in the middle being of a material different from that in the layers on both sides of it. Another problem treated is that of a cantilever beam with a single force acting at the free end; a more accurate solution than the classical one of Saint Venant is obtained.

J. J. Stoker (New York, N. Y.).

Iguchi, Shikazo. Die Eigenschwingungen und Klangfiguren der vierseitig freien rechteckigen Platte. Proc. Phys.-Math. Soc. Japan (3) **23**, 733–757 (1941).

The first 23 normal frequencies and the 35 corresponding normal modes of vibration of the square elastic plate with free edges are computed with great accuracy by developing in infinite series in the standard way. A set of infinitely many linear equations each containing infinitely many unknowns must be solved. This was the problem chosen by W. Ritz [Ann. Physik (4) **28**, 737–786 (1909)] on which to test the power of the energy method now called the Rayleigh-Ritz method. The author claims that his solutions are the first rigorous ones and that those of Ritz are not rigorous since the set of functions used by Ritz to form a minimizing sequence does not satisfy all of the boundary conditions. However, the reviewer feels that this criticism is not justified since the boundary conditions in the present case are the "natural" boundary conditions for the variational problem and hence need not be satisfied by the approximating functions. However, the author's solutions may well converge more rapidly than those of Ritz, although it should be said that Ritz was able to calculate quite closely the experimental frequencies obtained by Chladni. The author presents photographs of sand figures showing the nodal lines for various normal modes which are in good agreement with his calculated results. No comparison of the calculated frequencies with experimentally observed values is given.

J. J. Stoker (New York, N. Y.).

Seth, B. R. Stability of rectilinear plates. J. Indian Math. Soc. (N.S.) **10**, 13–16 (1946).

The author observes the well-known fact that the stability investigation of a thin elastic plate under plane hydrostatic compression, and the investigation of the vibrations of a membrane under uniform tension, lead (for corresponding boundaries) to identical eigenvalue problems.

G. F. Carrier (Providence, R. I.).

Seth, B. R. Finite longitudinal vibrations. Proc. Indian Acad. Sci., Sect. A. **25**, 151–152 (1947).

Assuming a nonlinear stress-strain law, the author reduces the problem of describing the longitudinal waves in a string to the problem of describing long waves in a canal [see Lamb, Hydrodynamics, 5th ed., Cambridge University Press, 1930, p. 241]. *G. F. Carrier* (Providence, R. I.).

Haag, J. Sur les vibrations des fils élastiques et des fils parfaits. Ann. Sci. École Norm. Sup. (3) **63** (1946), 185–254 (1947).

A detailed study of many problems of the vibration of elastic lines (i.e., slender elastic filaments) and of inextensible cords. The author considers lateral, longitudinal and torsional vibrations of "lines" which are straight, circular, helical, constrained to a spherical surface, etc. The kinematic and force analyses are carried out in detail for small displacements. Many tables of natural frequencies corresponding to particular boundary conditions and initial configurations are provided. *G. F. Carrier.*

Batdorf, S. B., Schildcrout, Murry, and Stein, Manuel. Critical shear stress of long plates with transverse curvature. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1346**, 21 pp. (2 plates) (1947).

Batdorf, S. B., Schildcrout, Murry, and Stein, Manuel. Critical combinations of shear and longitudinal direct stress for long plates with transverse curvature. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1347**, 27 pp. (9 plates) (1947).

Batdorf, S. B., Stein, Manuel, and Schildcrout, Murry. Critical shear stress of curved rectangular panels. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1348**, 18 pp. (11 plates) (1947).

Batdorf, S. B. A simplified method of elastic-stability analysis for thin cylindrical shells. I. Donnell's equation. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1341**, 38 pp. (12 plates) (1947).

Batdorf, S. B., Stein, Manuel, and Schildcrout, Murry. Critical stress of thin-walled cylinders in torsion. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1344**, 21 pp. (5 plates) (1947).

Batdorf, S. B., Stein, Manuel, and Schildcrout, Murry. Critical combinations of torsion and direct axial stress for thin-walled cylinders. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1345**, 28 pp. (8 plates) (1947).

Krall, G. Moltiplicatore critico λ_{cr} d'una distribuzione di carico su una volta autoportante. I. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **1**, 1281–1294 (1946).

Goodey, W. J. The stresses in a circular fuselage. J. Roy. Aeronaut. Soc. **50**, 833–871 (1946).

This paper analyzes the stresses and deformations in elastic orthotropic circular cylindrical shells, with loads applied by means of circular reinforcing rings. The analysis is simplified by the (reasonable) assumption of a certain limiting-case orthotropy which reduces the order of the basic differential equation compared to the corresponding order for the isotropic shell. [The assumptions and results are similar in essence to those of N. J. Hoff [J. Appl. Mech. **11**, A-235–A-239 (1944)] and of L. Beskin [J. Appl. Mech. **13**, A-137–A-147 (1946)]. The methods of derivation in these earlier papers are different and somewhat more compact.] *E. Reissner* (Cambridge, Mass.).

Hardtwig, Erwin. Über die Wellenausbreitung in einem visko-elastichen Medium. Z. Geophys. **18**, 1–20 (1943).

Starting from the well-known theory of body and surface waves in a perfectly elastic isotropic medium and considering some of the infinite number of possible relationships between the stress and strain tensors when viscosity is present, the author chooses to discuss in detail the simple case of linear dependence of the stress components both on the strain components and on their first derivatives with respect to time. The constants in the viscosity term of the wave equation are thus assumed to be proportional to those in the elastic term. All particular solutions indicate damping dependent on the viscosity and increasing with the square of the frequency. They also indicate dispersion; but they do not show the well-known increase of period with distance. The author concludes that these constants have no direct relation to the coefficient of viscosity as defined in hydromechanics, that the proportionality factor between the constants in the elastic terms and those in the flow terms must be quite small and that a solution of the boundary value problem is required to show the observed lengthening of period with distance. *J. B. Macelwane.*

Scholte, J. G. On the propagation of seismic waves. Nederl. Akad. Wetensch., Proc. **49**, 1115–1126 (1946).
Scholte, J. G. On the propagation of seismic waves. II. Nederl. Akad. Wetensch., Proc. **50**, 10–17 (1947).

The author calculates the motion caused by a normal pressure which acts uniformly on the surface of a monostratified medium, it being assumed that the pressure is periodic in time. The corresponding surface motion may be calculated by the saddle point method. It is remarked that it is also possible to study the propagation of seismic waves in any horizontally stratified medium. *A. E. Heins.*

Ramspeck, A. Reine Longitudinal- und Transversalwellen im elastisch-homogenen Halbraum. Z. Geophys. **18**, 21–27 (1943).

The author assumes as valid the usual wave equations for an isotropic and homogeneous elastic medium. Applying the boundary condition that the stress must vanish in the boundary, and assuming a general D'Alembert solution for a radial disturbance at the origin in the boundary plane, the author evaluates the potential functions and finds that the differential equation for pure longitudinal waves yields no useful solution. He finds that pure transverse waves can only be generated by a disturbance in the boundary if the displacement vector is parallel to the boundary and perpendicular to the direction of propagation. He concludes that it is impossible to produce only longitudinal waves by detonation of explosives at the surface of the ground. A part of the energy will always go into transverse and Rayleigh waves. The fraction of the energy of the explosion that goes into longitudinal waves that are first to arrive increases with depth of burial of the charge. *J. B. Macelwane.*

Gogoladze, V. Reflection and refraction of non-stationary elastic waves. C. R. (Doklady) Acad. Sci. URSS (N.S.) **49**, 322–325 (1945).

Two semi-infinite homogeneous isotropic elastic media with different Young's and shear moduli and densities are separated by the y-plane. The author considers the propagation across the plane boundary of an incident transverse elastic plane wave with normal in the z-plane and issuing in the medium (1), which has the smaller transverse wave velocity. The incident wave shape in the direction of propagation is arbitrary except for reasonable continuity conditions (suitable form of boundedness of the second derivative). For component phase velocities in (1) larger than the greater of the two longitudinal propagation velocities, the solution is classical [Green, Mathematical Papers, London, 1871] and consists of a quadruplet of waves, i.e., longitudinal and transversal incident, reflected, and refracted components in (1) and (2) with the same wave shape functions as the incident one, obtained by straightforward satisfaction of the boundary conditions demanding continuity of stresses and displacements along $y=0$. There are three other cases involving component phase velocities smaller than the larger one of the two longitudinal wave velocities, which are separated, (a) by the smaller of these and (b) by the (larger) transversal velocity of (2). These cases partly involve inhomogeneous waves. The solutions are obtained by splitting the real shape function of the incident wave into two analytic conjugate components (in analogy with the vector representation used in electrical engineering) regular in the upper and lower half planes, respectively; this is permissible under the assumed continuity conditions. The complex components are amenable to the same simple procedure of satisfaction of the boundary conditions as was the original function in the Green case, and the results appear in similar form but involve, partly, the real components of the analytic function as inhomogeneous waves. Physically, this form can again be interpreted as a wave quadruplet but with both real and complex angles of reflection and refraction.
H. G. Baerwald (Cleveland, Ohio).

Gogoladze, V. On Rayleigh boundary waves. C. R. (Doklady) Acad. Sci. URSS (N.S.) **49**, 400–403 (1945).

The same physical conditions are assumed as in the paper reviewed above. The author investigates the conditions for existence of free vibrations or Rayleigh waves along the plane boundary. If $a_i > b_i$; $i=1, 2$, denote the longitudinal and transversal propagation velocities in media 1 and 2, respectively, $\sigma \equiv \mu_2/\mu_1$ the ratio of the shear moduli and ϑ the phase velocity of the Rayleigh wave, the Rayleigh equation $R(\vartheta)=0$ is obtained as a polynomial in ϑ^2 and the four branch terms $(1-\vartheta^2/a_i^2)^{\frac{1}{2}}$, $(1-\vartheta^2/b_i^2)^{\frac{1}{2}}$ with powers up to ϑ^4, linear in each of the branch terms and containing a_i, b_i and $1-\sigma$ as coefficients. Complex roots correspond to inhomogeneous, real ones to homogeneous waves, i.e., only the latter case represents (free) Rayleigh waves. Detailed investigation of the doubly connected two-sheeted Riemann surface of $R(\vartheta)$ determines the class of domains of elastic media where Rayleigh waves are possible as $R(b_2)>0$ for $b_2<b_1$ or $R(b_1)>0$ for $b_1>b_2$. Putting $\vartheta=\xi+i\eta$, $\xi\equiv\lambda(1-\sigma)$, $\delta\equiv\rho_2/\rho_1$ (density ratio), the identity $\delta=\sigma b_1^2/b_2^2$ defines the straight line $\eta=(b_1^2/b_2^2)(1-\frac{1}{2}\xi)$, $b_1^2/b_2^2>1$ in the ϑ-plane. The points on this line situated outside the parabola $P(\xi,\eta)=0$, where P is defined by $R(\vartheta)\equiv P(\vartheta)-iQ(\vartheta)(\vartheta^2/b_2^2-1)^{\frac{1}{2}}$, determine the Rayleigh waves. The analytic form of these boundary waves is given. *H. G. Baerwald* (Cleveland, Ohio).

Gogoladze, V. General formulae for the reflexion and refraction of non-stationary elastic waves. C. R. (Doklady) Acad. Sci. URSS (N.S.) **49**, 479–481 (1945).

This is an analytic summary of the two papers reviewed above. The four wave potentials are given in closed form as sums over the Rayleigh (surface) and four space components (longitudinal or transversal in either medium), all, in general, complex; the propagation velocity of the Rayleigh wave if homogeneous, i.e., physically existent, is always smaller than the smallest one of the space waves. An expression for the energy flow S and its x- and y-components in terms of these wave potentials is obtained. If x denotes the direction of real phase velocity while the y-component may be complex (inhomogeneous), it is shown that the sign of S_x is always uniform while that of S_y may change in time and space. [This is obvious for physical reasons.] *H. G. Baerwald* (Cleveland, Ohio).

MATHEMATICAL PHYSICS

Optics, Electromagnetic Theory

***Morais, Cesare. Metodo generale per lo sviluppo e lo studio delle aberrazioni nei sistemi ottici.** Atti Secondo Congresso Un. Mat. Ital., Bologna, 1940, pp. 628–645. Edizioni Cremonense, Rome, 1942.

For the description of a ray which undergoes refraction at a surface of revolution S, the author uses three reference planes perpendicular to the axis of the system. Two of these planes (p, q) may be regarded as arbitrary; the third (v) passes through the vertex V of S. Let (P_1, Q_1) be the intersections of the ray with the planes (p, q), and (P, Q) the projections of (P_1, Q_1) on the axis of the system. Let the lines (PQ_1, P_1Q) cut the plane v at the points (A, B). Then the distances VA, VB, together with the angle between PP_1 and QQ_1, describe the ray completely, except for a rotation about the axis of the system. The author uses these ray-coordinates to discuss refraction at a sphere and at an aspheric surface, with the corresponding aberrations, and also the refraction of a wave; the case of refraction through a pair of surfaces is also discussed. The author does not compare his method with other methods which have been used [cf. J. P. C. Southall, The Principles and Methods of Geometrical Optics, Macmillan, New York, 1913, p. 304; M. Herzberger, Strahlenoptik, Springer, Berlin, 1931, p. 45]. *J. L. Synge* (Pittsburgh, Pa.).

Friedlander, F. G. Geometrical optics and Maxwell's equations. Proc. Cambridge Philos. Soc. **43**, 284–286 (1947).

The paper gives a generalization of the methods which derive geometrical optics from wave optics. The difference from the ordinary treatment consists in the fact that the author does not restrict himself to the consideration of a single plane wave as solution of the wave equation, but considers an asymptotic series which formally forms a solution of Maxwell's equation. He finds that for large values of $k=1/\lambda$ the first member of the asymptotic series alone gives a representation leading to the fundamental law of geometric optics including the energy considerations. The methods of the author are valid in a general inhomogeneous anisotropic medium and permit finding the laws of polarization in such a medium. *M. Herzberger.*

Abelès, Florin. Formules relatives à une lame mince transparente, baignée par deux milieux transparents, dans le cas de la réflexion totale. C. R. Acad. Sci. Paris **224**, 1494–1496 (1947).

Würschmidt, José. Laws of reflection in a moving mirror for corpuscles and for photons in empty space and in a refractive medium. Univ. Nac. Tucumán. Revista A. **5**, 321–333 (1946). (Spanish)

Duffieux, P. Michel. Remarques sur les phénomènes de diffraction. Ann. Physique (12) **2**, 95–132 (1947).

In the first section of the paper the author points out certain difficulties arising in the discusson of Fraunhofer diffraction phenomena by means of the complex displacement functions $F(x, y)$ and $G(u, v)$, which describe respectively the characters of the waves passing through the (x, y)-plane and of the corresponding diffracted waves at infinity and are connected by the reciprocal formulae

$$(1)\qquad F(x,y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}G(u,v)e^{2\pi i(ux+vy)}du\,dv,$$

$$(2)\qquad G(u,v)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F(x,y)e^{-2\pi i(ux+vy)}dx\,dy.$$

Here the direction cosines u, v automatically satisfy the conditions $u^2+v^2\leqq 1$ and therefore the function $G(u, v)$ is to be set equal to zero outside this part of the domain of integration in (1). It then follows that $F(x, y)$ cannot also satisfy the condition of being zero everywhere outside a certain finite region in the (x, y)-plane. That is, equations (1), (2) cannot hold for the "widely diffracted light" issuing from an aperture in an opaque screen.

To meet this difficulty, the author proposes to treat the directly observable function $G(u, v)$ as given and to regard

$F(x, y)$ as determined by (1). In the cases of importance to instrumental optics, where $G(u, v)$ is small except in the neighbourhood of $(u, v)=(0, 0)$, the author considers that the value of $F(x, y)$ so obtained will represent to a sufficient approximation the waves issuing from the finite exit-pupil of the system, although it does not strictly satisfy the condition of vanishing everywhere outside this exit-pupil.

In the second section an attempt is made, in the case where $F(x, y)$ is zero outside a certain finite area of the (x, y)-plane, to separate "edge effect" from "aperture effect" on the basis of the equations

$$\frac{\partial F}{\partial x}=2\pi i\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}uG(u, v)e^{2\pi i(ux+vy)}dudv,$$

$$\frac{\partial F}{\partial y}=2\pi i\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}vG(u, v)e^{2\pi i(ux+vy)}dudv$$

and their formal corollaries

$$(3)\qquad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\frac{\partial F}{\partial x}\right|^2 dxdy=4\pi^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}u^2|G(u, v)|^2dudv,$$

$$(4)\qquad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\frac{\partial F}{\partial y}\right|^2 dxdy=4\pi^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}v^2|G(u, v)|^2dudv.$$

Since (3), (4) are not mathematically valid with the definitions adopted for $F(x, y)$ and $G(u, v)$, the attempt is not completely successful.

In the third and fourth sections of the paper, attempts are made to discuss the diffraction of light-corpuscles and the question of coherence from a similar standpoint.

E. H. Linfoot (Bristol).

Macfarlane, G. G. A variational method for determining eigenvalues of the wave equation applied to tropospheric refraction. Proc. Cambridge Philos. Soc. **43**, 213–219 (1947).

The author's summary is as follows. A simple and direct variational method is described for finding both complex and real eigenvalues of the wave equation of anomalous propagation in a horizontally stratified atmosphere. It may be looked upon as an extension of Rayleigh's method to complex eigenvalues. In this paper it is illustrated by an example, taken from the duct theory of super-refraction, in which the refractive index of the air varies with height according to a power law. Numerical agreement in the values for the lowest order eigenvalues with those obtained by the differential analyser is better than $\frac{1}{2}\%$.

L. Hulthén (Lund).

Costa de Beauregard, Olivier. Quelques calculs d'électromagnétisme relativiste. Ann. Physique (12) **1**, 522–537 (1946).

This paper consists of three unrelated parts, each having to do with the interpretation of four-dimensional tensors occurring in the Maxwell electromagnetic theory. In the first part the author defines the field strength tensor for a point charge of strength Q as Q times the field strength tensor per unit charge.

In part 2 the author attempts to relate the heat generated by a current in a conductor, thought of as a gas without pressure, with the change in proper mass of the conductor.

In part 3 the author discusses the asymmetrical Maxwell-Minkowski stress-energy tensor for the electromagnetic field in an arbitrary medium. He proposes a new definition for this tensor defined in terms of the vector potential and the current vector. However, this tensor is not gauge invariant. The author discusses the anti-symmetric part of this tensor and states that it does not correspond to any physical quantity.

A. H. Taub (Princeton, N. J.).

Baudot, Jacques. Sur la représentation matricielle des équations de Maxwell. C. R. Acad. Sci. Paris **224**, 735–737 (1947).

Maxwell's equations in free space and in the absence of charges may be written $(c^{-1}\partial_t-\partial_i\mathfrak{A}_i)\mathbf{F}=0$, where $\mathbf{F}$ is the tensor of the electromagnetic field components and the $\mathfrak{A}_i$ are real symmetric matrices with four indices, only eight of the sixteen equations being distinct. Properties of the matrices $\mathfrak{A}_i$ are given; the operator in the brackets is replaced by a Hermitian one involving only matrices which anticommute (as do the $\mathfrak{A}_i$ but not the unit matrix). Connexions are established with a theory of a particle of zero rest-mass due to G. Petiau [Thèse, Paris, 1936].

C. Strachan (Aberdeen).

Svartholm, Nils, and Siegbahn, Kai. An inhomogeneous ring-shaped magnetic field for two-directional focusing of electrons and its application to β-spectroscopy. Ark. Mat. Astr. Fys. **33A**, no. 21, 28 pp. (1947).

In a magnetic field of rotational symmetry there exist, under certain conditions, a discrete number of stationary paths of charged particles. These paths are circles normal to the axis of symmetry and centered on this axis. Conditions for a stable stationary motion on such circles have been given by Wallauschek [Z. Physik **117**, 565–574 (1941); these Rev. **8**, 363]. Charged particles which originate in the neighborhood of stationary paths will remain in this neighborhood provided that their original direction includes a sufficiently small angle with the stationary path. The authors show that neighboring bundles of this type have focusing properties with astigmatic character in general. In special magnetic fields, however, stigmatic focusing can be obtained. The derivation of these fields is the aim of the paper. It is also shown how a magnetic field of the desired type can be realized practically and used with advantage in the analysis of particle radiation from active elements (β-spectroscopy). A description is given of a spectrograph installation designed on the basis of the above principle.

R. K. Luneberg (New York, N. Y.).

Svartholm, Nils. The resolving power of a ring-shaped inhomogeneous magnetic field for two-directional focusing of charged particles. Ark. Mat. Astr. Fys. **33A**, no. 24, 10 pp. (1947).

In the paper reviewed above, special magnetic fields have been derived which provide stigmatic focusing of charged particles in the neighborhood of stable stationary paths. The accuracy of the image formation, i.e., the resolving power, depends on the aperture of the bundle and is limited by aberrations of particles which include a finite initial angle with the central stationary path. In the previous paper these aberrations were determined rigorously for rays which lie in the plane of the central path. The general case of other field forms and of rays of arbitrary direction is discussed in the present paper. The equations of motion are solved approximately by a method of successive approximations and the size and shape of the image of a point object are calculated.

R. K. Luneberg (New York, N. Y.).

Pinney, Edmund. Electromagnetic fields in a paraboloidal reflector. J. Math. Phys. Mass. Inst. Tech. 26, 42–55 (1947).

The author finds the electromagnetic field in an infinite paraboloid of revolution of infinite conductivity which acts as a reflector. The field is generated by radiating dipoles, and three different cases are given: (i) a dipole at the focus and orientated parallel to the axis; (ii) a dipole at the focus and orientated perpendicular to the axis of the reflector; (iii) a dipole in the same position as in (ii), but backed by a "dummy reflector" which is a secondary dipole placed, in front of the first, on the axis. Paraboloidal coordinates are used and the solutions are obtained in the form of infinite series of appropriate solutions of the wave equation in paraboloidal coordinates. These solutions have been investigated in a previous paper by the author [same J. 25, 49–79 (1946); cf. these Rev. 7, 442]. [The corresponding acoustical problem was investigated, also in paraboloidal coordinates but otherwise in a different manner, by H. Buchholz, Ann. Physik (5) 42, 423–460 (1942); these Rev. 5, 249.]

A. Erdélyi (Pasadena, Calif.).

***Aharoni, J. Antennae. An Introduction to Their Theory.** Oxford, at the Clarendon Press, 1946. viii+265 pp. $8.50.

Recent developments in antenna theory are brought together and made readily available in this concise book. Chapter I, "Antennae and boundary-value problems," is introductory, and considers various applications of Maxwell's equations used in the later chapters. It includes such topics as the retarded potentials, properties of plane and spherical waves, the electromagnetic field of an infinitely long wire and that of two infinitely long coaxial cones, and concludes with a discussion of free and forced oscillations in spheres and prolate spheroids.

Chapter II, "Antennae and integral equations," is mainly concerned with the Hallén integral equation for the current in a straight thin antenna. The general problem is approached by first considering the quasi-stationary state, when it is shown that, by assuming uniform currents and neglecting radiation, the ordinary circuit equations can be derived from Maxwell's equations. Next at higher frequencies the electromagnetic fields of electric and magnetic dipoles are derived, still assuming uniform currents but no longer neglecting radiation. Finally the Hallén integral equation is obtained by assuming stationary lines of current flow parallel to the axis of the wire. Hallén's solution is described and the general properties of systems of a number of thin wires are obtained. This chapter also includes a discussion of receiving antennas, of Carter's circuit relations, of the polar diagrams for antenna arrays and of the effect of the earth on antenna radiation.

Chapter III, "Antennae as wave guides," deals with Schelkunoff's approach to antenna theory, in which the antenna is represented as a transmission line carrying a principal mode and terminated into a proper terminal impedance. For a symmetrical biconical antenna the line is uniform; this case is considered in detail. Antennas of other shapes are then treated as slightly nonuniform transmission lines.

M. C. Gray (New York, N. Y.).

Müller-Strobel, J., und Patry, J. Der Empfangsdipol. Ableitung einer Formel für den Antennenstrom. Schweiz. Arch. Angew. Wiss. Tech. 12, 201–213 (1946).

The solution follows the conventional method of power expansions in terms of the parameter Ω, based on Hallén's theory. Various simplifications accrue by assuming the aerial length to be small compared with the wavelength. The tangential field along the aerial is likewise expanded and the three first terms are obtained in closed form as a practical approximation. Error limitation to about 5% is realized assuming validity of the inequality $L=2\pi l/\lambda\leq .19$. Various integrals obtained in this procedure are expanded in powers of L and the series may be broken off after L^4 on the ground of this inequality. A compact final expression for the center feed current is based on these approximations; restriction to the zero order term gives a rough estimate. [A short summary of the contents and results was given in Helvetica Phys. Acta 17, 127–132 (1944); these Rev. 6, 55.]

H. G. Baerwald.

Van Vleck, J. H., Bloch, F., and Hamermesh, M. Theory of radar reflection from wires or thin metallic strips. J. Appl. Phys. 18, 274–294 (1947).

From a mathematical standpoint the problem of radar reflection from a metallic strip (or its equivalent cylindrical wire) is essentially that of determining the current in a receiving wire antenna when a linearly polarized wave is incident at an angle θ with the axis of the wire. Two expressions for this current are developed by the authors; the more important (method B) is derived from the integral equation for the current by a somewhat modified form of Hallén's method. In the alternative method (method A) a form of current function similar to that obtained by B is assumed and the arbitrary functions involved are determined by equating the work done on the wire to the total radiated flux. While the calculation of the current by A is somewhat simpler it is open to the objection that for long wires (several wave lengths) the resonance peaks are located at integral values of $4l/\lambda$ instead of at slightly longer wave lengths.

As a measure of the efficiency of the wire for radar reflection the authors define the cross-section σ as equal to 4π times the ratio of the power reflected backwards per unit solid angle to the incident power density. For a bundle of randomly oriented wires the averaged cross-section $\bar{\sigma}=\int\int\sigma\sin\theta\, d\theta d\varphi$ is used. Asymptotic formulas for these cross-sections for long wires are obtained and curves are drawn of their variation as functions of l/λ for different values of the ratio of wire length to radius. Some preliminary experimental results are also included.

M. C. Gray (New York, N. Y.).

Slater, J. C. Microwave electronics. Rev. Modern Physics 18, 441–512 (1946).

In this review article the author has attempted to put the theory of resonant cavities into a unified setting. In order to do so he has considered not only the electronics of the resonator, but in addition the resonant circuit, consisting of resonant cavity and attached loads, and the reaction of this circuit back on the electronic motion. This has been accomplished by means of the development of a circuit theory of resonant cavities and of wave guides, which form the leads to these cavities, based on the theory of orthogonal functions. The orthogonal functions are chosen to satisfy boundary conditions which are conveniently close to the boundary conditions of the actual problem. In the subsequent expansion of the solution in terms of these orthogonal functions, one term can usually be singled out as the dominant term of the expansion near any given resonant frequency. The author is able to give a rather complete

qualitative description of the electronic resonator. Besides furnishing a unifying basis for microwave electronics, this development has been of practical value in the design and development of magnetrons. The article concludes with an application of these general methods to the klystron and the magnetron. *R. S. Phillips* (Los Angeles, Calif.).

Kovalenkov, V. I. Separation of a complex system of equations into a combination of simple mutually independent systems or equations. Avtomatika i Telemehanika 7, 7–14 (1946). (Russian)

Two parallel transmission lines carrying current of the same frequency $\omega/2\pi$ give rise to a system of linear homogeneous partial differential equations with constant coefficients for currents and voltages with distance x and time t as the independent variables. Solutions are developed by the standard method of separation of variables, the unknowns $u, \cdots$ being assumed of the form $U(x)e^{i\omega t}, \cdots$, leaving an ordinary system with x as the independent variable. *S. Lefschetz* (Princeton, N. J.).

Castelluccio, D. Studio analitico dei problemi delle linee e dei filtri elettrici. Ist. Lombardo Sci. Lett. Cl. Sci. Mat. Nat. Rend. (3) 4(73), 175–226 (1940).

By considering effects of reflection, interference, etc., the author formulates the solution of the long line problem as an infinitely iterated integral. Without considering convergence and by observing the form of the first few iterations, he expresses his results in the usual Bessel function formulas. The latter part of the paper is devoted to filters. *N. Levinson* (Cambridge, Mass.).

Herreng, Pierre, et Ville, Jean. Sur la stabilité des réseaux linéaires. Rev. Gén. Électricité 54, 93–96 (1945).

A derivation of the Nyquist criterion is given. *N. Levinson* (Cambridge, Mass.).

***De Stefano, Alberto. Sui teoremi di reciprocità nella radiotelegrafia.** Atti Secondo Congresso Un. Mat. Ital., Bologna, 1940, pp. 794–802. Edizioni Cremonense, Rome, 1942.

The author shows that the reciprocity theorems of Carson [Bell System Tech. J. 3, 393–399 (1924)] and of Ballantine [Proc. I. R. E. 17, 929–951 (1929)] differ from each other only because the former neglected a term in his derivation. The author generalizes the theorem in question to nonperiodic electromagnetic systems in which the current **C** and the electric field **E** are related by the equation

$$\mathbf{C}=\sigma\mathbf{E}+\epsilon(\partial\mathbf{E}/\partial t)+\int_0^t f(t-x)\mathbf{E}(x)dx,$$

where σ and ϵ depend only on the point in the space and f is considered as a known function. The author's method is based on an application of the Laplace transform to the above and to the Maxwell equations. *I. Opatowski.*

Ferraro, V. C. A. The radial stability of the geomagnetic ring-current. Terr. Magnetism 51, 547–555 (1946).

This paper completes and substantiates the conclusions of an earlier paper by S. Chapman and the author [Terr. Magnetism 46, 1–6 (1941); these Rev. 3, 255] in which the effect of the displacement currents was neglected. The exact solution obtained in this paper proves that, though the current sheet is in reality unstable, in all cases of practical interest the rate at which instability sets in is so slow that the sheet may well be considered to be stable. The main conclusions reached in the earlier paper are therefore still valid. The rate of decay of the current sheet deduced in this paper is of the order of the observed rate of decrease of intense and moderate storm-effects on D_m. *E. Kogbetliantz* (New York, N. Y.).

Quantum Mechanics

Dedebant, G. Les schémas aléatoires devant la relativité restreinte. Portugaliae Phys. 2, 149–202 (1946).

The author defines a stochastic wave as the expression $\psi=A\cos(\Omega t-Mx+\Phi)$, where (Ω, M) is a pair of variates (in general, interdependent), while A and Φ are variates which are independent of one another and of (Ω, M), Φ being uniformly distributed on $(0, 2\pi)$; t and x are the time and 1-space coordinates. The autocorrelation $r(\tau, k)$ of ψ is $\cos(\Omega\tau-Mk)$ averaged over all variates A, Φ, Ω, M, and $\tau=t_2-t_1$, $k=x_2-x_1$. If ψ is to be relativistically meaningful, $r(\tau, k)$ must be invariant when x, t (and hence τ, k) are subjected to the Lorentz transformation. This leads to rules of transformation for the various moments of (Ω, M). All these are materially extended by replacing the Lorentz transformation by the author's "extended Lorentz transformation", which is what the former becomes on replacing $(1-\beta^2)^{\frac{1}{2}}$ by $(|1-\beta^2|)^{\frac{1}{2}}$, $\beta=v/c$. The author regards this extended transformation as more appropriate to the stochastic theory since the latter deals, not with coincidences of particles, but with correlations, and whether these are measured at two points (t_1, x_1) and (t_2, x_2) joined by a space-like vector (τ, k) or a time-like one is inessential: the extended Lorentz transformation allows one type of vector to be transformed into the other. The group of extended transformations contains the usual one as a subgroup.

The invariants of stochastic waves, as well as of random distributions of events in space-time, and of electromagnetic fields, under the extended Lorentz transformations are studied, and, after making certain physical identifications, some of the elementary formulas of quantum mechanics are obtained. The familiar wave-corpuscle duality is developed, not only in the classical case of corpuscles slower than light, but in the case of corpuscles faster than light, whose associated waves are slower than light and might correspond to light passing through refractive media. Under the extended Lorentz transformation, all this assumes a unified form. The latter part of the paper is devoted to speculations on the elementary particles of matter.

Apart from the simplest formulas, no quantum mechanics is considered: states (vectors in function space), their laws of change with time (Schroedinger's wave equation), the quantum theory of measurement, etc., are entirely left out of account. Indeed, the author states that his paper is intended chiefly to call the attention of physicists to the possible uses of his stochastic methods, rather than to provide any definitive results. *B. O. Koopman.*

Shu, Seyuan. The foundations and philosophical implications of wave mechanics. Canadian J. Research. Sect. A. 25, 96–117 (1947).

Arnous, Edmond, et Colombo, Serge. Sur l'indépendance stochastique des observables. C. R. Acad. Sci. Paris 224, 376–377 (1947).

This note extends a previous result of one of the authors [Arnous, same C. R. 219, 389–391 (1944); these Rev. 7,

271]. It gives a condition, in terms of the orthogonality of certain subspaces, that two observables are stochastically independent. In the previous note the operators associated with the variables were assumed to be bounded; in the present note they may be unbounded. *O. Frink.*

Cazin, Michel, et Viard, Jeannine. Les trièdres non-commutatifs non-holonomes en cinématique opératorielle. C. R. Acad. Sci. Paris **224**, 452–454 (1947).

This extends some of the results of a previous note [Cazin, same C. R. **222**, 992–994 (1946); these Rev. **8**, 121], concerning a type of noncommutative operational kinematics, to the nonholonomic case. *O. Frink* (State College, Pa.).

Viard, Jeannine. Petits mouvements autour d'une position d'équilibre stable en mécanique ondulatoire. Disquisit. Math. Phys. **3**, 131–140 (1943).

Prigogine, I., et Garikian, G. Sur le calcul des niveaux énergétiques par la méthode de Wentzel-Kramers-Brillouin et son application à l'hydrogène liquide. J. Phys. Radium (8) **7**, 330–332 (1946).

Dirac, Paul A. M. Developments in quantum electrodynamics. Communications Dublin Inst. Advanced Studies. Ser. A. no. **3**, 33 pp. (1946).

The author reviews the role of redundant variables (variables without physical meaning which do not enter the wave equations except as parameters in the wave functions) in quantum mechanics. He points out that if they are used in the wave functions then one must introduce a weight function in these variables in order to obtain a physical interpretation of the wave functions.

The quantum equations describing a system of charged particles and an electromagnetic field have been previously discussed by the author in terms of redundant variables [same Communications. Ser. A. no. 1 (1943); these Rev. **7**, 100]. The weight function used there was chosen so that certain integrals would converge at the price of introducing negative probabilities. In this paper the author shows how the weight function may be determined in terms of the initial configuration of the field and proposes to use this weight function and abandon the perturbation method for solving the wave equations. However no general method is given to replace the perturbation method. He also points out that an extra condition is needed to correspond to the classical condition that the final acceleration is zero but states that "it is not at all clear what form this condition should take."

The paper concludes with a discussion of a property of wave functions describing a radiation field when the position of the photon is known. The property in question is the occurrence of singularities in the wave function considered as a function of a complex variable.

A. H. Taub (Princeton, N. J.).

Pirenne, Jean. Le champ propre et l'interaction des particules de Dirac suivant l'électrodynamique quantique. Arch. Sci. Phys. Nat., Geneva (5) **28**, 233–272 (1946).

The object of the present investigation is the study of the proper field of the electron and the examination of the mutual interaction of two Dirac electrons in terms of it. The paper is stated to be the first of three parts. The first section deals mainly with the recapitulation of classical electrodynamics in a form convenient for translation to quantum theory. Maxwell's equations are written in terms of the Fourier components of the vector potential and then expressed in Hamiltonian form. The interaction between classical electrons is then calculated in order to illustrate the method of attack to be used later and Darwin's expression for the interaction energy between two electrons moving with nonrelativistic velocities is obtained. Next the interaction energy between two electric and magnetic dipoles is derived. In the second section Dirac's theory is taken up. After the usual description of the theory, the transition charge and current densities are written in a form which shows the correspondence with the classical expression for the charge and current produced by the motion of a particle with charge and magnetic and electric dipole moments.

S. Kusaka (Princeton, N. J.).

Sokolow, A. On the polarization of electron waves. Acad. Sci. USSR. J. Phys. **9**, 363–372 (1945). [MF 15122]

The polarization of electron waves is investigated for the cases of reflection from a potential barrier and of scattering by a center of force. For the two-dimensional problem of the reflection and refraction of plane electron waves by a step-function potential whose magnitude is small compared to the kinetic energy of the electron, it is shown that both the reflected and the refracted waves are unpolarized if the incident wave is unpolarized. For the problem of scattering by a center of force, a general formula is first derived by using Dirac's time-dependent perturbation theory and it is then applied to the case where the force center possesses electric charge and magnetic moment. Since the existence of polarization can only be detected by analysis of the scattered waves by another scattering process, the distributions for double scattering by two centers of force are calculated. It is shown that in the first approximation polarization terms occur only if the scattering centers possess both electric and magnetic moment, neither the charge or the moment by itself being sufficient. When the calculation is carried out to the second approximation, pure Coulomb interaction gives rise to polarization effects and the formula obtained here coincides with that obtained by Mott previously in a different way. *S. Kusaka* (Princeton, N. J.).

Sokolow, A. On the polarization of electron waves. Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. **16**, 3–14 (1946). (Russian. English summary)

A translation is reviewed above.

Drăganu, Mircea. Sur l'énergie propre de l'électron et l'introduction d'une longueur fondamentale en mécanique quantique. Disquisit. Math. Phys. **2**, 109–126 (1942).

This paper described a method of removing the infinite self-energy in the usual electron theory by the introduction of a fundamental length. The hypothesis is made that the electron is not localizable at a definite point at a definite instant and its interaction with the electromagnetic field is supposed to extend over a finite region of space. This domain of uncertainty is taken as a four-dimensional region of Euclidean space; the reviewer does not follow the reasoning by which the transformation to Minkowski space is made. The self-energy of the electron satisfying the Dirac equation is carried out by the usual perturbation method and the value $137mc^2/(2\pi\gamma)$ is obtained, where γ is a certain numerical constant. The commutators of the electromagnetic field variables are also studied and a modified form of the Jordan-Pauli generalized delta function is obtained.

S. Kusaka (Princeton, N. J.).

Chang, T. S. A note on relativistic second quantization. Proc. Cambridge Philos. Soc. **43**, 183–195 (1947).

Relativistic second quantization for the Fermi-Dirac case along lines similar to Dirac's treatment [Proc. Roy. Soc. London. Ser. A. **180**, 1–40 (1942); these Rev. **5**, 277] of the Bose-Einstein case. *A. Pais* (Princeton, N. J.).

Harish-Chandra. On relativistic wave equations. Physical Rev. (2) **71**, 793–805 (1947).

The relativistic invariance of a first order wave equation of the type $i\beta^k\partial_k\psi+\chi\psi=0$, where β^k $(k=0, 1, 2, 3)$ are square matrices operating on ψ which is itself a one-column matrix and where χ is a constant determining the mass of the particle, is examined. It is shown that the problem of finding suitable representations of the β-matrices is intimately connected with the structure of their enveloping algebra. In particular, the center of this algebra can contain only elements of a very restricted type and there are strong conditions on the spurs of the β-matrices and their multiple products. Conversely, if these conditions are satisfied, the equation is relativistically invariant. The general theory is illustrated by considering the Dirac and the Duffin-Kemmer equations in s dimensions and some general remarks are made concerning the theory of particles with higher spin. *S. Kusaka* (Princeton, N. J.).

Bodiou, Georges. Sur une condition mathématique permettant de limiter à quatre le nombre de composantes de la fonction d'onde de Dirac. C. R. Acad. Sci. Paris **224**, 721–723 (1947).

Remarks on the four anti-commuting matrices which appear in Dirac's equation of the electron. *V. Bargmann* (Princeton, N. J.).

Kwal, Bernard. Sur les équations d'onde non linéaires de la théorie quantique de l'électron. C. R. Acad. Sci. Paris **224**, 1099–1100 (1947).

The author believes that, in order to overcome the present difficulties of quantum theory, it may be necessary to introduce nonlinear wave equations derived from a variational principle. The Lagrangian must be so chosen that in a first approximation Dirac's or Schroedinger's equations are obtained. Two simple examples are mentioned. *V. Bargmann* (Princeton, N. J.).

Kwal, Bernard. Théorie non linéaire du photon et du méson. Modification de la loi de Yukawa. C. R. Acad. Sci. Paris **224**, 1207–1208 (1947).

Kwal, Bernard. Sur la mécanique ondulatoire des corpuscules élémentaires. Arch. Sci. Phys. Nat., Geneva **26**, 135–152 (1944); **27**, 5–25, 95–121, 167–190, 211–221 (1945). [MF 16530]

An extensive investigation of possible wave equations for particles of arbitrary spin and both finite and zero rest mass. Starting from the wave equations for particles of spin $\frac{1}{2}$, the author first derives "primary equations" by the method of exterior multiplication. Combining these in different ways, he obtains wave equations for particles of higher spin, of which the equations previously proposed by various authors (including, of course, Dirac's equations) are special cases. The equations are stated in matrix and in spinor form and their relativistic invariance is proved. The solutions corresponding to plane waves are studied in some detail. *V. Bargmann* (Princeton, N. J.).

Pais, A. On the theory of elementary particles. Verh. Nederl. Akad. Wetensch. Afd. Natuurk. Sect. 1. **19**, no. 1, 91 pp. (1947).

This paper surveys the quantum mechanics of elementary particles, as point sources, interacting with various fields; the problem of infinite self-energies is treated by the cancellation of the infinite contributions of two or more fields. Chapter I surveys the calculation of self-energies, on one-particle and hole theory, for a particle of spin $\frac{1}{2}\hbar$ in interaction with Einstein-Bose fields: f, g interactions involving field potentials, derivatives of field potentials, only, are used. Weisskopf's treatment of interaction with the electromagnetic (e) field [Physical Rev. (2) **56**, 72–85 (1939)] is discussed: tables summarize the divergences (first order perturbation theory) for scalar, vector, pseudoscalar, pseudovector fields. Higher order self-energies are discussed where possible. Chapter II deals with the electron. Classical unitary and quantum nonunitary (no definition of mass in terms of field quantities) theories are contrasted [e.g. Stückelberg, Helvetica Phys. Acta **14**, 51–80 (1941)]. Subtractive fields are rejected. The divergence due to the e field is cancelled by that due to one scalar f field with hole theory, but not less than two g fields. Chapter III treats the nucleon and nuclear forces. Proton and neutron are different states of one particle with respect to e and f charge. Convergence relations cannot be satisfied by meson fields singly or in symmetrical vector-pseudoscalar or neutral scalar-pseudovector combination. A scalar F field is proposed [Hulthén, Kungl. Fysiografiska Sällskapets i Lund Förhandlingar [Proc. Roy. Physiog. Soc. Lund] **14**, no. 2 (1944)] with range shorter than the usual meson field range. In chapter IV further consequences of the f field are considered, e.g., relating to the S levels of hydrogen and to cosmic ray phenomena. *C. Strachan* (Aberdeen).

Flint, H. T. A study of the nature of the field theories of the electron and positron and of the meson. Proc. Roy. Soc. London. Ser. A. **185**, 14–34 (1946). [MF 15183]

Field theories of the electron and meson are developed in a five-dimensional formalism similar to Kaluza's theory of electromagnetism. It has already been shown by various authors that this five-dimensional representation has certain formal advantages, such as unifying into a single tensor the energy-momentum tensor and the current density vector of the usual theory. In the present paper explicit expressions for such tensors are written down for the electron satisfying the Dirac equation and for the meson satisfying the Proca equation. Then in the latter case nuclear sources of the meson field are introduced and it is shown that, when the coordinator of the fifth dimension is assumed to vary sinusoidally, the ratio g_1/g_2 of the strengths of the nuclear sources is determined by the meson mass. *S. Kusaka.*

Caldirola, Piero. Integrazione delle equazioni del campo mesonico. Ricerca Sci. **13**, 195–198 (1942).

A purely classical treatment. The equation

$$\Delta\varphi-c^{-2}\partial^2\varphi/\partial t^2-k^2\varphi=f(x, y, z, t)$$

is solved using operational methods. The solutions are generalized retarded potentials, expressed as integrals containing the Bessel function J_1. By way of illustration the author considers the case of a single point-source at rest. *L. Hulthén* (Lund).

Heitler, W. A theorem in the charge-symmetrical meson theory. Proc. Roy. Irish Acad. Sect. A. 51, 33–39 (1946).

Kemmer's formulation of the exchange forces between nucleons due to meson exchange leads to invariance under a group of transformations; this group may be regarded as rotations in the "charge space." The nuclear potentials themselves are invariant under such rotations. The paper investigates the effects of rotations in the charge space on the scattering of mesons by nucleons. It is found that the scattering cross-sections for the scattering of mesons on nucleons depend on the state of charge of the nucleon. A nontrivial relation between the cross-sections for scattering of mesons of different charges by nucleons in both states of charge is derived from invariance considerations in the charge space. *L. Jánossy* (Dublin).

Wentzel, G. Recent research in meson theory. Rev. Modern Physics 19, 1–18 (1947).

Gião, Antonio. Forces nucléaires, gravitation et électromagnétisme. Portugaliae Math. 5, 145–193 (1946).

This continues previous work by the same author [Portugaliae Phys. 2, 1–98 (1946); these Rev. 8, 121] and part of this is given again here at some length. Equations of motion for the elementary mass-corpuscles and for the elementary charge-corpuscles, the two not being necessarily coincident in space, are derived by integration of the divergences of the previously introduced tensors T_n^{ik}, U_n^{ik} for each corpuscle n. These tensors are each expressed as the sum of a part referring to the motion of the mean centre of the corpuscle and a part referring to the internal structure of the corpuscle, the former part in each case being identified with the usual tensor of general relativity, referring however to the internal and external metrics for the mass- and charge-corpuscles, respectively. For particles having both mass and charge it is supposed that the metric is given by a linear combination of the g_{ik}, ω_{ik}. For an approximately flat space-time these equations produce forces which are identified with the Lorentz force, the force of gravitation and nuclear forces. By a series of definitions, whose justification is not apparent to the reviewer, equations are derived having formal resemblance to those of a type of meson field. *C. Strachan* (Aberdeen).

Gião, António. Quelques propriétés des fonctions d'onde cosmologiques des particules élémentaires. Gaz. Mat., Lisboa 7, no. 30, 4–5 (1946).

This summarises work already reviewed [Portugaliae Phys. 2, 1–98 (1946); these Rev. 8, 121, and the paper reviewed above] and refers to forthcoming related work. *C. Strachan* (Aberdeen).

Murard, Robert. Les conditions de normalisation en théorie du corpuscule libre. C. R. Acad. Sci. Paris 224, 807–809 (1947).

Ginsburg, V., Landau, L., Leontovitsh, M., and Fock, V. On the failure of A. A. Vlasov's papers on generalized theory of plasma and theory of solid state. Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. 16, 246–252 (1946). (Russian. English summary)

The authors criticize Vlasov's papers in Bull. Acad. Sci. URSS. Sér. Phys. [Izvestia Akad. Nauk SSSR] 8, 248–266 (1944); Acad. Sci. URSS. J. Phys. 9, 25–40, 130–138 (1945); Uchenye Zapiski Moskov. Gos. Univ. Fizika 77, 3–29, 30–42 (1945); these Rev. 6, 222; 7, 104, 183.

Thermodynamics, Statistical Mechanics

Kryloff, N., et Bogolioùboff, N. Sur les équations de Focker-Planck déduites dans la théorie des perturbations à l'aide d'une méthode basée sur les propriétés spectrales de l'hamiltonien perturbateur. (Application à la mécanique classique et à la mécanique quantique.) Ann. Chaire Phys. Math. Kiev 4, 5–157 (1939). (Ukrainian and French)

This is the first paper of a series which proposes to establish the theory of perturbations and transitions of state upon a new and uniform basis, both in classical and quantum mechanics. The essential idea appears to be to avoid setting out with the assumption of a priori transition probabilities between the states of the unperturbed system (transitions induced by the perturbations) and instead to assume, in the perturbing term of the Hamiltonian, certain parameters having a given probability distribution. A method of successive approximations is set up for finding the perturbative effect upon the "constants" of the motion, the method involving averaging over the parameters in the perturbing term. After a complicated set of approximate computations, quantities are derived which play the rôles of transition probabilities; equations are obtained which correspond to the Focker-Planck equation and the conventional transition equations in quantum mechanics.

The example of the method worked out in this paper is that of a quasi-periodic system perturbed by an incoherent set of harmonically oscillating forces (e.g., a train of light waves, acting on dipoles), the latter having an approximately continuous (and limited) spectrum in the classical case. The random phases of these perturbing terms are the parameters over which the averaging cited above is performed. This is first worked out for a classical system; then, with the aid of von Neumann's density matrix, in the quantum case. In the latter, the result is compared with the conventional perturbation method, a critique of which is given. *B. O. Koopman* (New York, N. Y.).

Vonsovskiĭ, S. V. Derivation of fundamental kinetic equation in quantum mechanics. Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. 16, 908–918 (1946). (Russian)

An English translation appeared in Acad. Sci. USSR. J. Phys. 10, 367–376 (1946); these Rev. 8, 364.

***Rasetti, Franco. Les nouvelles statistiques en physique.** Proc. First Canadian Math. Congress, Montreal, 1945, pp. 219–232. University of Toronto Press, Toronto, 1946. $3.25.

Bohm, David, and Aller, Lawrence H. The electron velocity distribution in gaseous nebulae and stellar envelopes. Astrophys. J. 105, 131–150 (1947).

In this paper the Boltzmann equation is formulated for an assembly in which electrons, liberated by photo-electric ionizations of neutral hydrogen atoms, suffer elastic scattering by other electrons and ions, inelastic scattering with ions giving rise to metastable ions, and collisions of the second kind with metastable ions and protons, and are finally recaptured. The cross sections for the various processes considered are assembled and the Boltzmann equation is derived in the usual manner. The authors find that under the conditions prevailing in a planetary nebula "the velocity distribution is very close to Maxwellian. Because the average lifetime of an electron in the assembly is about 10 years

and because its energy is reshuffled nearly every second by electrolastic encounters, the deviations from the equilibrium Maxwellian distribution must be very small."

S. Chandrasekhar (Williams Bay, Wis.).

Urban, P. Beitrag zur intermediären Statistik. Akad. Wiss. Wien, S.-B. IIa. 152, 111–135 (1943).

The author calculates the thermodynamical properties of the degenerate and nondegenerate "Gentile-gas" for the relativistic case. The fluctuations are discussed on the same basis. *F. London* (Durham, N. C.).

Caldirola, Piero. Osservazioni sulle statistiche intermedie. Ricerca Sci. 12, 1020–1027 (1941).

Sommerfeld, Arnold. Die Quantenstatistik und das Problem des Heliums II. Ber. Deutsch. Chemisch. Ges. Abt. B. 75, 1988–1996 (1942).

Elementary discussion of the distribution function of the intermediate statistics of Gentile. *F. London.*

Schubert, Gerhard. Zur Bose-Statistik. Z. Naturforschung 1, 113–120 (1946).

The method of steepest descent of Fowler and Darwin is applied to the "intermediate" statistics of G. Gentile in order to obtain a rigorous estimate of the order of magnitude of the deviation of the Gentile statistics (for the case $d=N$) from the Bose-Einstein statistics. It is shown that the Gentile term is of the order of e^{-N} whereas in the ordinary Bose-Einstein formula terms of the order of $1/N$ are already neglected. It is accordingly inconsistent to supplement the ordinary Bose-Einstein formula by the Gentile term, at least as long as one neglects the terms of the order of $1/N$.

F. London (Durham, N. C.).

Gentile, Giovanni, J. Sopra il fenomeno della condensazione del gas di Bose-Einstein. Ricerca Sci. 12, 341–346 (1941).

The case of strong degeneracy of an ideal gas is calculated on the basis of the author's "intermediate" statistics [see Ist. Lombardo Sci. Lett. Cl. Sci. Mat. Nat. Rend. (3) 5(74), 133–137 (1941); these Rev. 8, 364] for the case that the maximum number d of particles per cell is equal to N, the total number of particles. It is shown that the resulting "condensation" (discontinuities of thermal quantities) is exactly the same as the one obtained on the basis of ordinary Bose-Einstein statistics [F. London, Physical Rev. 54, 947–954 (1938)], which is identical with the case $d\to\infty$ of the Gentile statistics. [Reviewer's remark. This result is not surprising since the thermodynamical discontinuities are defined only for the limit $N\to\infty$ in which case the two theories are identical.] *F. London* (Durham, N. C.).

Wergeland, H., and Hove-Storhoug, K. Gibbs' phase integral for separable systems. A contribution to the theory of Einstein's condensation phenomenon. I. Norske Vid. Selsk. Forh., Trondhjem 15 (1942), no. 34, 131–134 (1943).

Wergeland, H., and Hove-Storhoug, K. Calculation of the isotherm for a degenerate gas. A contribution to the theory of Einstein's condensation phenomenon. II. Norske Vid. Selsk. Forh., Trondhjem 15 (1942), no. 35, 135–138 (1943).

Wergeland, H., and Hove-Storhoug, K. On the equation of state for a degenerate gas. A contribution to the theory of Einstein's condensation phenomenon. III. Norske Vid. Selsk. Forh., Trondhjem 15 (1942), no. 47, 181–183 (1943).

The main purpose of these papers is to show "how neatly Einstein's conclusions [concerning the condensation phenomenon] can be derived by means of the canonical ensemble of Gibbs." The authors combine the expression for the free energy ψ, $\exp(-\psi/kT)=\mathrm{Spur}\exp(-H/kT)$, with a Dirac δ-function to obtain the approximate expression

$$\psi=-kT\log(2\pi i)^{-1}\int_{\sigma-i\infty}^{\sigma+i\infty}e^{-Nt+V\xi(3/2,\,t)/\lambda^3}dt,$$

where $\xi(s,t)=\sum_1^\infty t^s/n^s$ and $\lambda=h/(2\pi mkT)^{\frac{1}{2}}$. They then obtain Einstein's parametric representation of the equation of state but give a precise description of the condensation phenomenon by means of the contour integral for ψ.

C. C. Torrance (Annapolis, Md.).

Wergeland, Harald. Bose-Einstein condensation and the new statistics of G. Gentile. Norske Vid. Selsk. Forh., Trondhjem 17, no. 13, 51–54 (1944).

The restriction concerning the maximum number of particles per cell characteristic of the Gentile statistics is expressed by means of a Dirichlet factor. The free energy ψ is obtained in the form

$$e^{-\psi/\theta}=\frac{1}{\pi i}\frac{d}{dN}\int_{\sigma+i\infty}^{\sigma-i\infty}t^{-1}e^{-Nt}dt\prod_{s=0}^{\infty}\left\{\frac{1-\exp[(N+1)(t-\epsilon_s/\theta)]}{1-\exp(t-\epsilon_s/\theta)}\right\}.$$

Here $\sigma<0$ and the product is to be taken over the discrete energy values ϵ_s of a single molecule. *F. London.*

Wergeland, Harald. Bose-Einstein condensation and the new statistics of G. Gentile. II. Norske Vid. Selsk. Forh., Trondhjem 17, no. 15, 63–66 (1944).

It is shown that even if one neglects the quantization of the energy values assumed as discontinuous in the expression for the free energy in the note reviewed above and replaces sums by integrals one still obtains the correct zeropoint pressure and zeropoint energy of a degenerate Bose-Einstein gas, which are due to the molecules in the lowest quantum state. *F. London* (Durham, N. C.).

BIBLIOGRAPHICAL NOTES

***Problems of Mathematics.** Princeton University Bicentennial Conferences. Series 2, Conference 2. Princeton University, Princeton, N. J., 1947. 32 pp. (2 plates)

This booklet contains brief reports of the sessions of the conference held December 17–19, 1946.

***Bericht über die Mathematiker-Tagung in Tübingen vom 23. bis 27. September 1946.** Herausgegeben vom Mathematischen Institut der Universität Tübingen. 143 pp.

Contains 49 abstracts of papers presented at the conference.